Population Invariance Properties of Social and Economic Networks*

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This version: June 22, 2016

Abstract

This paper investigates how a pairwise stable network changes upon the arrival of a new agent. Comparing the new networks obtained for the society after the arrival of a new agent with the initially pairwise stable network, we propose four different population invariance properties of a network: *link invariance*, *distance invariance*, *connectedness invariance*, and *network invariance*. First, we show that pairwise stability is incompatible with *link invariance* under certain assumptions on the allocation rule. However, if we consider specific models, positive results can be obtained. For instance, in the symmetric connections model, pairwise stability implies *connectedness invariance*.

JEL classification: C70; D70; D85

Keywords: Network formation; variable population; pairwise stability; invariance property

1. Introduction

A standard model of network formation is concerned with the following situation: given a set of agents, two agents can add a link if they agree on it; any agent can sever a link if she wants to; and adding a link is beneficial but also costly. In many studies (Jackson and Wolinsky 1996; Watts 2001; Jackson and Watts 2002; and others), agents are assumed to behave myopically,

*We are grateful to Chris Chambers, Matt Jackson, Makoto Yano, and a referee for their comments. Chun's work was supported by the National Research Foundation of Korea Grant funded by the Korean Government (NRF-2013S1A3A2055391), and the Institute of Economic Research, Seoul National University.

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that is, agents do not consider how their behavior might affect others' decisions in the future.[1] An improving path is a sequence of adjacent networks that can be obtained when agents add or sever a link based on the myopic expectation, and a pairwise stable network is one in which no agent has an incentive to add or sever a link.[2] Jackson and Wolinsky (1996) introduce pairwise stability and analyze the relation between pairwise stability and efficiency. Watts (2001) studies a dynamic process of network formation in a specific model, the connections model. Jackson and Watts (2002) extend the dynamic network formation model to a general setting where agents can add or sever a link by mistake and investigate the implications of stochastic stability.

In this paper, we investigate how a pairwise stable network changes in the variation of the set of agents assuming the agents behave myopically.[3] More precisely, for a given set of agents, suppose that a pairwise stable network is initially constructed. If a new agent enters the network, then the initial network may not be pairwise stable anymore. A new network for the new set of agents will be constructed through an improving path, which leads to a pairwise stable network or to networks in a closed cycle, each of which can be viewed as a limiting network of the initial network. We investigate whether the relations between the initial agents remain unchanged in its limiting networks for the new society. Our study can be applied to analyze how a pairwise stable network changes upon the arrival of a new neighbor in a town.

We propose four different population invariance properties of a network which help to make a comparison between the initial network and its limiting networks. *Link invariance* requires that the direct relations between the initial agents remain unchanged. *Distance invariance* requires that the

[1] For studies on the farsighted behavior of agents in the context of networks, see Deroian (2001), Watts (2002), Dutta, Ghosal, and Ray (2005), Page and Wooders (2009), and Herings, Mauleon, and Vannetelbosch (2009).

[2] For other definitions of stability, see Dutta and Mutuswami (1997), Slikker and van den Nouweland (2001), Jackson and van den Nouweland (2005), Bloch and Jackson (2006), Galeotti et al. (2010), and Bramoulle, Kranton, and D'Amours (2014).

[3] There have been studies on dynamic network models with variable population. For instance, Jackson and Rogers (2007), Currarini, Jackson, and Pin (2009), and Bramoulle et al. (2012) examine the structure of steady-state networks with agents entering sequentially into a society. However, they do not investigate population invariance properties of a steady-state network.

distances between any two initial agents remain unchanged. *Connectedness invariance* requires that the connectedness between any two initial agents remain unchanged. Finally, *network invariance* requires that the initial network remain unaffected: there is no severance of the existing links and no addition of any new links. *Network invariance* implies *distance invariance*, which in turn implies both *link invariance* and *connectedness invariance*. For a connected network, *link invariance* implies *connectedness invariance*, but in general, there is no direct logical relation between these two properties.

First, we show that, for a class of allocation rules satisfying *component balance*, *weak link symmetry*, and *weak improvement*, pairwise stability is incompatible with *link invariance*. That is, there is a value function for which no pairwise stable network is *link invariant*. *Component balance* requires that if a value function is component additive, then the value of a component should be allocated to the agents in the component. *Weak link symmetry* requires that if the per capita value of two agents increases by adding a link between them and a new link is profitable to one of the two agents, then it should be profitable to the other agent. *Weak improvement* requires that, if two agents increase the value of a network by adding a link and this new link is beneficial to a third agent, then it should also be beneficial to at least one of the two agents adding the link. Although these axioms are satisfied by a class of allocation rules including the componentwise egalitarian rule, they are restrictive enough to give the negative result.

On the other hand, if we restrict our attention to specific models introduced in Jackson and Wolinsky (1996), we can obtain positive results. In the symmetric connections model, every pairwise stable network is *connectedness invariant*. If a pairwise stable network is complete, then it is *link invariant*. Furthermore, depending on the values of the parameters of the model, a pairwise stable network can be *distance invariant* or *network invariant*. In the coauthor model, the complete network is always pairwise stable and *link invariant*. Moreover, it is the only pairwise stable network satisfying *link invariance*.

2. A Model of Networks

We follow the terminology of Jackson and Wolinsky (1996) and Jackson and Watts (2002), but modify the notation to allow variations in the set of agents.

Agents

Let $\mathbf{N} \equiv \{1, 2, \ldots\}$ be a (finite or infinite) universe of "potential" agents. Let $\mathcal{N}$ be the family of nonempty finite subsets of $\mathbf{N}$, with elements denoted by N and N'.

Networks

The relations between agents are represented by a network in which each node is identified with the agent and each link captures the pairwise relation. For all $N \in \mathcal{N}$, let L^N be the set of all subsets of N of size 2. A *network* (or a *graph*) is $g \equiv (N, L)$ with $N \in \mathcal{N}$ and $L \subseteq L^N$. For all $N \in \mathcal{N}$, let $\mathcal{G}^N$ be the set of all networks for N, and $\mathcal{G} \equiv \cup_{N \in \mathcal{N}} \mathcal{G}^N$. For all $i, j \in N$, the (undirected) *link* between i and j is denoted by ij. For all $g = (N, L) \in \mathcal{G}$, if $ij \in L$, then nodes i and j are directly connected in g, and if $ij \notin L$, then nodes i and j are not directly connected in g. For all $g \in \mathcal{G}$, let $N(g)$ be the set of nodes in g and $L(g)$ be the set of links in g. For all $N \in \mathcal{N}$, let $e^N \equiv (N, \emptyset)$ be the *empty network* for N and $g^N \equiv (N, L^N)$ be the *complete network* for N. A network $g = (N, L)$ is a *singleton network* if $|N| = 1$.

For all $N \in \mathcal{N}$, let $N^c \equiv \mathbf{N} \setminus N$ be the set of agents not in N. For all $g = (N, L) \in \mathcal{G}$ and all $k \in N^c$, let $g \oplus k \in \mathcal{G}^{N \cup \{k\}}$ be the network obtained by adding a new agent k to N without affecting the set of links, that is, $g \oplus k \equiv (N \cup \{k\}, L)$. For all $g = (N, L) \in \mathcal{G}$ and all $i, j \in N$ with $ij \notin L$, let $g + ij = (N, L \cup \{ij\})$ be the network obtained by adding link ij to g. For all $g = (N, L) \in \mathcal{G}$ and all $i, j \in N$ with $ij \in L$, let $g - ij = (N, L \setminus \{ij\})$ be the network obtained by deleting link ij from g. If $g' = g + ij$ or $g' = g - ij$, then g and g' are *adjacent*.

A *chain* in $g = (N, L) \in \mathcal{G}$ connecting i_1 and i_n is a set of distinct nodes $\{i_1, i_2, \ldots, i_n\} \subseteq N$ such that $\{i_1 i_2, i_2 i_3, \ldots, i_{n-1} i_n\} \subseteq L$. If such a chain exists, then nodes i_1 and i_n are *connected*. A network $g = (N, L) \in \mathcal{G}$ is *connected* if for all $i, j \in N$ with $i \neq j$, there is a chain in g connecting i

and j. A network $g' = (N', L') \in \mathcal{G}$ is a *subnetwork* of $g = (N, L) \in \mathcal{G}$ if $N' \subseteq N$ and $L' \subseteq L$, written as $g' \subseteq g$. A connected subnetwork g' of $g \in \mathcal{G}$ is a *component* of g if for all $g'' \in \mathcal{G}$ with $g' \subseteq g'' \subseteq g$ and $g'' \neq g'$, g'' is not connected. In other words, a component of g is a maximal connected subnetwork of g. Note that an isolated node constitutes a component according to our definition. Let $C(g)$ be the set of all components of g.

For all $g = (N, L) \in \mathcal{G}$ and all $i, j \in N$ with $i \neq j$, if i and j are connected, then the (geodesic) *distance* between i and j in g, $d(i, j; g)$, is the smallest number of links between i and j. If not connected, $d(i, j; g) = \infty$. If $i = j$, $d(i, j; g) = 0$.

Value Functions and Allocation Rules

A *value function* is a function $v : \mathcal{G} \to \mathbb{R}$ which associates with each $g = (N, L) \in \mathcal{G}$ a value in $\mathbb{R}$. We normalize v by setting the value of a singleton component equal to zero. For all $N \in \mathcal{N}$, let $\mathcal{V}^N$ be the set of all value functions for N, and $\mathcal{V} \equiv \cup_{N \in \mathcal{N}} \mathcal{V}^N$.

An allocation rule, or a *rule,* is a function $Y : \mathcal{G} \times \mathcal{V} \to \cup_{N \in \mathcal{N}} \mathbb{R}^N$ which associates with each $(g, v) = ((N, L), v) \in \mathcal{G} \times \mathcal{V}$ a vector in $\mathbb{R}^N$. It allocates the value of a network to agents.

Pairwise Stability

Following Jackson and Wolinsky (1996), we assume that the formation of a link requires the consent of both agents, but the severance can be done unilaterally. Moreover, agents are assumed to behave myopically, that is, they do not forecast how adding or severing a link might affect the future formation of a network. A network is *pairwise stable* if no agent would benefit by severing an existing link and no two agents would benefit by adding a new link. Formally, a network $g = (N, L) \in \mathcal{G}^N$ *defeats* an adjacent network $g' = (N, L') \in \mathcal{G}^N$ if either (i) for some $ij \in L'$, $g = g' - ij$ and $Y_i(g, v) > Y_i(g', v)$, or (ii) for some $ij \in L$, $g = g' + ij$, $Y_i(g, v) > Y_i(g', v)$ and $Y_j(g, v) \geq Y_j(g', v)$. A network is *pairwise stable* with respect to v and Y if it is not defeated by any adjacent network.

Improving Paths and Cycles

As in Jackson and Watts (2002), an *improving path* from $g \in \mathcal{G}^N$ to $g' \in \mathcal{G}^N$ is a finite sequence of adjacent networks $\{g_1, \ldots, g_l\}$ with $g_1 = g$

and $g_l = g'$ such that, for any $t = 1, \ldots, l-1$, g_t is defeated by g_{t+1}. It captures a sequence of improvements in networks when agents form and sever links based on the myopic expectation.

For all $N \in \mathcal{N}$, a set of networks $C \subseteq \mathcal{G}^N$ is a *cycle* if for all $g, g' \in C$, there is an improving path from g to g'. A cycle C is a *closed cycle* if no network in C lies on an improving path leading to a network that is not in C.

Limiting Networks

For all $N \in \mathcal{N}$, $g' \in \mathcal{G}^N$ is a *limiting network* of $g \in \mathcal{G}^N$ if g' is a pairwise stable network or a network in a closed cycle that lies on an improving path from g.

From the proof of Lemma 1 in Jackson and Watts (2002), for any v and Y, each network that is not pairwise stable lies on an improving path to a pairwise stable network or to a network in a closed cycle. This establishes the existence of a limiting network. For all $g \in \mathcal{G}$, let $\mathcal{L}(g)$ be the set of all limiting networks of g.

For all g, $g' \in \mathcal{G}$, if g' is a limiting network of g, then g' is a pairwise stable network or a network in a closed cycle that lies on an improving path from g. If g' is pairwise stable, then there is an improving path from g to g' and there is no (further) improving path from g'. On the other hand, if g' is a network in a closed cycle, then there is an improving path from g to g' and there is no improving path from g' to a network outside of the closed cycle. Thus, g' is formed repeatedly along the closed cycle. In either case, the limiting network g' can be viewed as an outcome of a dynamic network formation process through an improving path starting from g. In our population invariance properties we will compare the structure of an initial pairwise stable network with all possible limiting networks obtained after the addition of a new agent.

Two Specific Models

Next we describe two specific models introduced in Jackson and Wolinsky (1996).

Symmetric Connections Model:

Agents communicate with those to whom they are connected. Let $0 < \delta < 1$ be the benefit of a direct link between any two agents and $c \geq 0$ be the cost of maintaining the link. Agents also benefit from indirect communications, but the benefit depreciates according to the distance $d(i, j; g)$ between i and j in g. For all $g \in \mathcal{G}$ and all $i \in N$, the utility of agent i in network g is given as

$$u_i(g) = \sum_{j \neq i} \delta^{d(i,j;g)} - \sum_{j:ij \in L(g)} c.$$

The value of network g is $v(g) = \sum_{i \in N} u_i(g)$ and the allocation rule $Y_i(g, v) = u_i(g)$.

Coauthor Model:

Agents spend time writing papers. Each link represents a collaboration between two agents. In network g, if $ij \in L(g)$, then agents i and j write a paper together. The number of papers involving agent i is denoted by $n_i(g) = |\{j | ij \in L(g)\}|$. For all $g \in \mathcal{G}$ and all $i \in N$, if $n_i(g) \geq 1$, the utility of agent i is given as

$$u_i(g) = \sum_{j:ij \in L(g)} \left[\frac{1}{n_i(g)} + \frac{1}{n_j(g)} + \frac{1}{n_i(g) n_j(g)} \right].$$

If $n_i(g) = 0$, then $u_i(g) = 0$. The value of network g is $v(g) = \sum_{i \in N} u_i(g)$ and the allocation rule $Y_i(g, v) = u_i(g)$.

3. Population Invariance Properties

A pairwise stable network $g \in \mathcal{G}^N$ with respect to v and Y is expected to remain unchanged when the set of agents, N, is fixed. However, if a new agent $k \in N^c$ enters the network, then g may not be pairwise stable. There may exist an improving path from $g \oplus k \in \mathcal{G}^{N \cup \{k\}}$ to another network $g' \in \mathcal{G}^{N \cup \{k\}}$. If g' is pairwise stable or in a closed cycle for the new society, then g' will be sustained or repeatedly formed. We investigate how the relations between the initial agents can be affected when a pairwise stable network changes due to the arrival of a new agent. We propose four different notions of population invariance properties of a network analyzing those relations.

Our first notion, *link invariance,* requires that upon the arrival of a new agent, the direct relations between all the initial agents would not be affected. In other words, all the initial links should not be severed and an additional link, if any, should connect the new agent with an initial agent.

Link Invariance: For all $N \in \mathcal{N}$, a pairwise stable network $g \in \mathcal{G}^N$ is *link invariant* with respect to v and Y if for all $k \in N^c$ and all $g' \in \mathcal{L}(g \oplus k)$, $L(g) = L(g') \cap L^N$.

Example 1: Consider the symmetric connections model with $N = \{1, 2, 3, 4\}$, $\delta = .9$, and $c = .2$. Let $g^* = (N, \{12, 23, 34\})$. Note that g^* is pairwise stable. Let $k \in N^c$ be a new agent and $g_0 = g^* \oplus k$. We can show that there are four improving paths from g_0 to pairwise stable networks, $\{g_0, g_0 + 2k\}$, $\{g_0, g_0 + 3k\}$, $\{g_0, g_0 + 1k, g_0 + 1k + 4k\}$, and $\{g_0, g_0 + 4k, g_0 + 1k + 4k\}$ (see Figure 1). Since $L(g^*) = L(g_0 + 2k) \cap L^N = L(g_0 + 3k) \cap L^N = L(g_0 + 1k + 4k) \cap L^N = \{12, 23, 34\}$, g^* is *link invariant.* $\square$

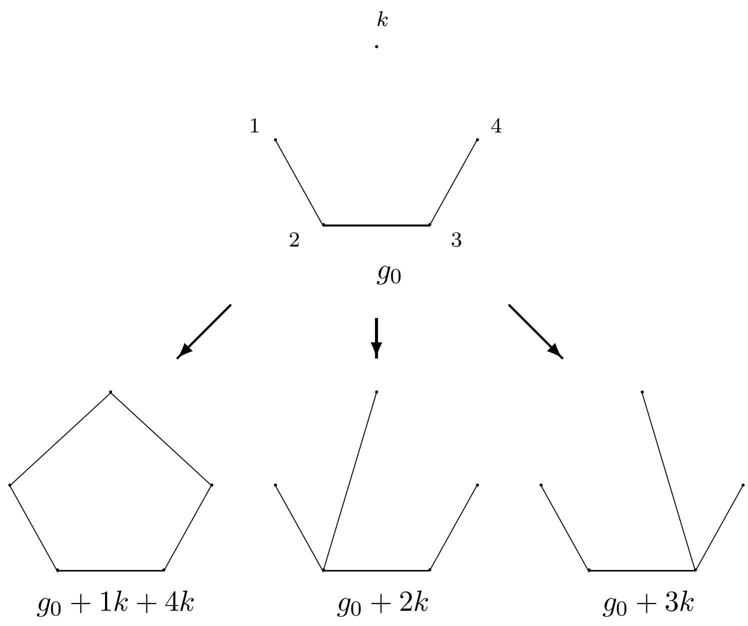

Figure 1 Link invariance

Link invariance is concerned with the *direct* relations between the initial agents. However, when a new agent enters the network, the *indirect* relations between the initial agents may have been affected even though their direct relations are unchanged. To analyze such a situation, we propose a stronger invariance property, *distance invariance,* which requires that the distance between any two initial agents be unaffected upon the arrival of a new agent.

Distance Invariance: For all $N \in \mathcal{N}$, a pairwise stable network $g \in \mathcal{G}^N$ is *distance invariant* with respect to v and Y if for all $k \in N^c$, all $g' \in \mathcal{L}(g \oplus k)$, and all $i, j \in N$, $d(i, j; g) = d(i, j; g')$.

Note that any *distance invariant* network is *link invariant.* However, the converse does not hold. In Example 1, for any $k \in N^c$, $d(1, 4; g^*) = 3 \neq 2 = d(1, 4; g_0 + 1k + 4k)$, so that the distance between 1 and 4 has been decreased from 3 to 2. Even though g^* is *link invariant,* it is not *distance invariant.*

Example 2: Consider the symmetric connections model with $N = \{1, 2, 3, 4\}$, $\delta = .9$, and $c = .5$. Let $g^* = (N, \{12, 23, 34\})$. Once again, note that g^* is pairwise stable. Let $k \in N^c$ be a new agent and $g_0 = g^* \oplus k$. For any $l \in N$, $\{g_0, g_0 + lk\}$ is an improving path and $g_0 + lk$ is pairwise stable. Moreover, there does not exist any other improving path. Since for all $l \in N$ and all $i, j \in N$, $d(i, j; g^*) = d(i, j; g_0 + lk)$, g^* is *distance invariant.* □

Our third invariance property, *connectedness invariance,* requires that if any two initial agents are connected before the arrival of a new agent, then they should remain connected afterwards, and if they are not connected before, then they should remain unconnected afterwards. In other words, it requires that the connectedness between the initial agents would not be affected upon the arrival of a new agent. For all $g = (N, L) \in \mathcal{G}$ and all $i \in N$, let $N_i(g) \equiv \{j \in N | d(i, j; g) < \infty\}$ be the set of agents connected with agent i in g.

Connectedness Invariance: For all $N \in \mathcal{N}$, a pairwise stable network $g \in \mathcal{G}^N$ is *connectedness invariant* with respect to v and Y if for all $k \in N^c$, all $g' \in \mathcal{L}(g \oplus k)$, and all $i \in N$, $N_i(g) = N_i(g') \setminus \{k\}$.

In Examples 1 and 2, it can easily be checked that the g^*'s are *connectedness invariant.*

Our last notion, *network invariance,* requires that the arrival of a new agent would not affect the initial network at all: the new agent does not make any link with the initial agents and the existing links between the initial agents are not affected. It can easily be shown that *network invariance* implies the other three invariance properties.

Network Invariance: For all $N \in \mathcal{N}$, a pairwise stable network $g \in \mathcal{G}^N$ is *network invariant* with respect to v and Y if for all $k \in N^c$, $g \oplus k \in \mathcal{G}^{N\cup\{k\}}$ is pairwise stable.

Example 3: Consider the symmetric connections model with $N = \{1, \ldots, 16\}$, $\delta = .9$, and $c = 1$. Jackson and Wolinsky (1996, p.52) show that the tetrahedron is pairwise stable. Let g be the tetrahedron. Let $k \in N^c$ be a new agent. Since $c > \delta$, each $i \in N$ is worse off by adding link ik to $g \oplus k$. Since g is pairwise stable for N, no initial agent has an incentive to sever or add a link with any other initial agent. Therefore, g is *network invariant.* □

Remark 1: As noted earlier, *network invariance* implies *distance invariance*, which in turn implies both *link invariance* and *connectedness invariance*. For a network with one component, *link invariance* implies *connectedness invariance*. However, for a network with more than one component, there is no direct logical relation between these two invariance properties.[4]

4. Results

We investigate whether a pairwise stable network can be *population invariant* on some class of allocation rules.

[4]Here we show by counterexamples that there is no direct logical relation between *link invariance* and *connectedness invariance* for a network with more than one component. First, a connectedness invariant network may not be link invariant. For example, in the proof of Theorem 1, $g = g^N - ij$ is connectedness invariant but not link invariant. Second, a link invariant network may not be connectedness invariant. Let $N = \{1, 2\}$. If $g_1 = (N, \{12\})$, $v(g_1) = -1$, and Y is the component-wise egalitarian rule, then $g_0 = (N, \emptyset)$ is pairwise stable. Now suppose that agent 3 enters the problem. Let $v(g_2) = 2$ for $g_2 = (\{1, 2, 3\}, \{13\})$, $v(g_3) = 6$ for $g_3 = (\{1, 2, 3\}, \{13, 23\})$, and all other networks yield zero value. Then g_3 is the only pairwise stable network (thus, it is the only limiting network from $g_0 \oplus 3$). Since $L(g_0) = \emptyset = L(g_3) \cap L^N$, g_0 is link invariant. However, $N_1(g_0) = \emptyset$ and $N_1(g_3) = \{1, 2, 3\}$, which together imply that g_0 is not connectedness invariant.

First, we introduce three axioms of an allocation rule. A value function v is *component additive* if for all $N \in \mathcal{N}$ and all $g \in \mathcal{G}^N$, $v(g) = \sum_{h \in C(g)} v(h)$. Our first axiom, *component balance*, requires that if a value function is component additive, then the value of a component should be allocated to the agents who belong to the component.

Component Balance: For all $N \in \mathcal{N}$, all $v \in \mathcal{V}^N$, all $g \in \mathcal{G}^N$, and all $h \in C(g)$, if v is component additive, then $v(h) = \sum_{i \in N(h)} Y_i(g, v)$.

For all $N \in \mathcal{N}$, all $g = (N, L) \in \mathcal{G}^N$ and all $i \in N$, let $h_i \in C(g)$ be the component of g containing i. For $h_i = (N_i, L_i)$ and $h_j = (N_j, L_j)$, if $h_i \neq h_j$, then $h_i + h_j = (N_i \cup N_j, L_i \cup L_j)$, and if $h_i = h_j$, then $h_i + h_j = (N_i, L_i)$. For all $N \in \mathcal{N}$, all $v \in \mathcal{V}^N$, all $g \in \mathcal{G}^N$ and all $h \in C(g)$, let $p(h, v) = v(h)/|N(h)|$ be the per capita value of component h with respect to value function v. For all $N \in \mathcal{N}$, all $v \in \mathcal{V}^N$ and all $g = (N, L) \in \mathcal{G}^N$, the *component-wise egalitarian rule* Y^{CE} is defined as follows: If v is component additive, then for all $i \in N$, $Y_i^{CE}(g, v) = p(h_i, v)$. If v is not component additive, then for all $i \in N$, $Y_i^{CE}(g, v) = v(g)/|N|$.

Our second axiom, *weak link symmetry*, requires that if the per capita value of two agents increases by adding a link between them and a new link is profitable to one of the two agents, then it should be profitable to the other agent. It was introduced by Dutta, van den Nouweland, and Tijs (1998) in the context of communication games. It is much weaker than *fairness* (Myerson 1977), which requires that a new link should affect the two agents forming the link by the same amount.

Weak Link Symmetry: For all $N \in \mathcal{N}$, all $v \in \mathcal{V}^N$, all $g = (N, L) \in \mathcal{G}^N$, and all $i, j \in N$ with $ij \notin L$, if $p(h_i + h_j + ij, v) \geq \max\{p(h_i, v), p(h_j, v)\}$, then $Y_i(g + ij, v) > Y_i(g, v)$ implies $Y_j(g + ij, v) > Y_j(g, v)$.

Improvement, also introduced by Dutta, van den Nouweland, and Tijs (1998) in the context of communication games, requires that the formation of a new link cannot benefit an agent who is not involved in the formation of the link without benefitting at least one of the two agents involved in the link. Our third axiom, *weak improvement*, requires that *improvement* should hold at least when the formation of a new link increases the network value.

Weak Improvement: For all $N \in \mathcal{N}$, all $v \in \mathcal{V}^N$, all $g = (N, L) \in \mathcal{G}^N$, and all $i, j \in N$ with $ij \notin L$, if $v(g+ij) > v(g)$, and for some $k \in N \setminus \{i, j\}$, $Y_k(g+ij, v) > Y_k(g, v)$, then $Y_i(g+ij, v) > Y_i(g, v)$ or $Y_j(g+ij, v) > Y_j(g, v)$.

Although these three axioms are satisfied by a class of allocation rules including the component-wise egalitarian rule, they are restrictive enough to give our main negative result.

Theorem 1: *There is no allocation rule which satisfies component balance, weak link symmetry, and weak improvement, and such that for all* $N \in \mathcal{N}$ *with* $|N| \geq 3$ *and all component additive* $v \in \mathcal{V}^N$*, at least one pairwise stable network is link invariant.*

Remark 2: It can be shown that our four requirements, the three axioms and *link invariance,* are independent of each other. That is, if we drop one of the four requirements, we can have a positive result. (The examples are available upon request.)

Now we turn our attention to specific models. In the symmetric connections model, all pairwise stable networks are *connectedness invariant.* If a pairwise stable network is complete, then it is *link invariant.* Furthermore, depending on the values of the parameters, a pairwise stable network can be *distance invariant* or *network invariant.*

Theorem 2: *In the symmetric connections model:*

(i) *Every pairwise stable network is connectedness invariant.*

(ii) *If a pairwise stable network is complete, then it is link invariant.*

(iii) *If* $c < \delta - \delta^2$*, every pairwise stable network is distance invariant.*

(iv) *If* $c > \delta$*, every pairwise stable network is network invariant.*

In the coauthor model, the complete network is always pairwise stable and *link invariant*. Moreover, it is the only pairwise stable network satisfying *link invariance.*

Theorem 3: *In the coauthor model, a pairwise stable network $g \in \mathcal{G}$ is link invariant if and only if g is the complete network.*

Remark 3: From Remark 1, *distance invariance* implies *link invariance.* In addition, for the complete network, the converse is also true. Therefore, in Theorem 3, *link invariance* can be replaced by *distance invariance.* Also, it can be shown that *link invariance* in Theorem 3 can be replaced by *connectedness invariance.* Since its proof can be obtained by mimicking the proof of Theorem 3, it is omitted. We note that a pairwise stable network in the coauthor model may not be *connectedness invariant* in contrast to the symmetric connections model.

5. Conclusion

This paper studies a situation in which only one agent enters into a pairwise stable network. In other words, we consider two pairwise stable networks merging into one when one of the two networks is a singleton. It would be interesting if we could develop a general theory describing what happens if two pairwise stable networks merge. For instance, suppose that a tetrahedron is pairwise stable in the symmetric connections model. When a single agent is added, she remains isolated, and the tetrahedron is network invariant. However, if another tetrahedron is added, a new link may be formed across the two tetrahedra.

Another interesting question is to ask what happens to a pairwise stable network if an agent leaves. The answer depends on the role of the leaving agent in the network. For example, suppose that a star network is pairwise stable in the symmetric connections model. By the symmetry of the model, it does not matter who enters the network as a new agent. However, a network obtained after deleting the central agent may be different from a network obtained after deleting a peripheral agent.

We hope to address these issues in our future research.

Appendix

Now we present the proofs for all theorems.

Proof of Theorem 1: Let $N \in \mathcal{N}$ be such that $|N| \geq 3$. Let Y be a rule satisfying *component balance*, *weak link symmetry*, and *weak improvement.* We show that there is a value function for which any pairwise stable network is not *link invariant.* Let $g \in \mathcal{G}^N$ and $\alpha > 0$. For all $h \in C(g)$, define $\bar{v}^\alpha$ as follows:

$$\bar{v}^\alpha(h) = \begin{cases} 0 & \text{if } h = g^{N(h)} \text{ and } |N(h)| \geq 3, \\ \alpha \cdot |L(h)| \cdot |N(h)| & \text{otherwise.} \end{cases}$$

For all $g \in \mathcal{G}^N$, define v^α as follows:

$$v^\alpha(g) = \sum_{h \in C(g)} \bar{v}^\alpha(h).$$

Note that v^α is component additive. Also, note that if $v^\alpha(g+ij) > v^\alpha(g)$, then $p(h_i + h_j + ij, v) \geq \max\{p(h_i, v), p(h_j, v)\}$. To see this, let i, $j \in N$ be such that $ij \notin L(g)$ and h_i, $h_j \in C(g)$. If $v^\alpha(g+ij) > v^\alpha(g)$ and $h_i \neq h_j$, then $p(h_i + h_j + ij, v^\alpha) = \alpha \cdot |L(h_i + h_j + ij)|$, $p(h_i, v^\alpha) = \alpha \cdot |L(h_i)|$ and $p(h_j, v^\alpha) = \alpha \cdot |L(h_j)|$. On the other hand, if $v^\alpha(g+ij) > v^\alpha(g)$ and $h_i = h_j$, then $p(h_i+h_j+ij, v^\alpha) = \alpha \cdot |L(h_i+ij)|$ and $p(h_i, v^\alpha) = p(h_j, v^\alpha) = \alpha \cdot |L(h_i)|$. In either case, we have $p(h_i + h_j + ij, v^\alpha) \geq \max\{p(h_i, v^\alpha), p(h_j, v^\alpha)\}$ and therefore, *weak link symmetry* can be applied between $g + ij$ and g.

Step 1: *Let $i, j \in N$ be such that $ij \notin L(g)$. If $v^\alpha(g+ij) > v^\alpha(g)$, then $Y_i(g+ij, v^\alpha) > Y_i(g, v^\alpha)$ and $Y_j(g+ij, v^\alpha) > Y_j(g, v^\alpha)$.*

Proof. Let $i, j \in N$ be such that $ij \notin L(g)$ and $v^\alpha(g+ij) > v^\alpha(g)$. First, suppose that for some $k \in N \setminus \{i, j\}$, $Y_k(g+ij, v^\alpha) > Y_k(g, v^\alpha)$. By *weak improvement*, $Y_i(g+ij, v^\alpha) > Y_i(g, v^\alpha)$ or $Y_j(g+ij, v^\alpha) > Y_j(g, v^\alpha)$. Since $v^\alpha(g+ij) > v^\alpha(g)$, by *weak link symmetry*, we have both $Y_i(g+ij, v^\alpha) > Y_i(g, v^\alpha)$ and $Y_j(g+ij, v^\alpha) > Y_j(g, v^\alpha)$. Next, suppose that for all $k \in N \setminus \{i, j\}$, $Y_k(g+ij, v^\alpha) \leq Y_k(g, v^\alpha)$. Since *component balance* implies that $\sum_{i \in N} Y_i(g+ij, v^\alpha) = v^\alpha(g+ij) > v^\alpha(g) = \sum_{i \in N} Y_i(g, v^\alpha)$, we must have either $Y_i(g+ij, v^\alpha) > Y_i(g, v^\alpha)$ or $Y_j(g+ij, v^\alpha) > Y_j(g, v^\alpha)$. Since $v^\alpha(g+ij) > v^\alpha(g)$, by *weak link symmetry*, we have both $Y_i(g+ij, v^\alpha) > Y_i(g, v^\alpha)$ and $Y_j(g+ij, v^\alpha) > Y_j(g, v^\alpha)$.

Step 2: *If network g is obtained by deleting more than one link from the complete network g^N with $|C(g)| = 1$, then for all $i, j \in N$ with $ij \notin L(g)$, g is defeated by $g + ij$.*

Proof. Let g be a network obtained by deleting more than one link from g^N with $|C(g)| = 1$. By the definition of v^α, for all $i, j \in N$ with $ij \notin L(g)$, $v^\alpha(g+ij) > v^\alpha(g)$. By Step 1, $Y_i(g+ij, v^\alpha) > Y_i(g, v^\alpha)$ and $Y_j(g+ij, v^\alpha) > Y_j(g, v^\alpha)$. Therefore, g is defeated by $g + ij$.

Step 3: *If g is a network with $|C(g)| > 1$ and $i, j \in N$ belong to different components of g, then g is defeated by $g + ij$.*

Proof. Let g be a network such that $|C(g)| > 1$. Let $h_i, h_j \in C(g)$ be the components that contain i and j respectively. Let $h_i = (N_i, L_i)$, $h_j = (N_j, L_j)$, and $h = (N_i \cup N_j, L_i \cup L_j \cup \{ij\})$. Since all the components other than h, h_i, and h_j are not changed, we can compare the values of g and $g + ij$ by focusing only on the values of h, h_i, and h_j. First, suppose that both h_i and h_j are singletons, which implies that $\bar{v}^\alpha(h_i) = \bar{v}^\alpha(h_j) = 0$. Since h is the complete network with two nodes, $\bar{v}^\alpha(h) = 2 \cdot \alpha$, so that $\bar{v}^\alpha(h) > \bar{v}^\alpha(h_i) + \bar{v}^\alpha(h_j)$. Next, suppose that at least one of h_i and h_j has more than one node. Since h is obtained by adding link ij connecting h_i and h_j, h is not the complete network for $N_i \cup N_j$. Therefore, $\bar{v}^\alpha(h) > \bar{v}^\alpha(h_i) + \bar{v}^\alpha(h_j)$. In both cases, $\bar{v}^\alpha(h) > \bar{v}^\alpha(h_i) + \bar{v}^\alpha(h_j)$, which implies that $v^\alpha(g + ij) > v^\alpha(g)$. By Step 1, $Y_i(g+ij, v^\alpha) > Y_i(g, v^\alpha)$ and $Y_j(g+ij, v^\alpha) > Y_j(g, v^\alpha)$. Therefore, g is defeated by $g + ij$.

Step 4: *If $g = g^N$, then for some $ij \in L(g)$, g is defeated by $g - ij$.*

Proof. Let $g = g^N$. Since $|N| \geq 3$, $v^\alpha(g) = 0$. Since *component balance* implies that $\sum_{i \in N} Y_i(g, v^\alpha) = v^\alpha(g) = 0$, for some $i \in N$, $Y_i(g, v^\alpha) \leq 0$. For such $i \in N$, let $g_0 = (N, \{lm \in L^N | l \neq i \text{ and } m \neq i\})$ be the network obtained by deleting all links of agent i from g. Since i is an isolated node in g_0, by *component balance*, $Y_i(g_0, v^\alpha) = 0$. Let $N \setminus \{i, j\} = \{l_1, \ldots, l_{|N|-2}\}$ and $g_1 = g_0 + il_1$, $g_2 = g_1 + il_2$, $\ldots$, $g_{|N|-2} = g_{|N|-3} + il_{|N|-2}$. Note that $g_{|N|-2} = g - ij$. Since $v^\alpha(g_0) < v^\alpha(g_1) < \cdots < v^\alpha(g_{|N|-2}) = v^\alpha(g - ij)$, by Step 1, $Y_i(g_0, v^\alpha) < Y_i(g_1, v^\alpha) < \cdots < Y_i(g_{|N|-2}, v^\alpha) = Y_i(g - ij, v^\alpha)$. Since $Y_i(g, v^\alpha) \leq 0 = Y_i(g_0, v^\alpha)$, $Y_i(g, v^\alpha) < Y_i(g - ij, v^\alpha)$. Therefore, g is defeated by $g - ij$.

Step 5: *A network g is pairwise stable if and only if for some $i, j \in N$, $g = g^N - ij$.*

Proof. We prove the "only if" part. Let g be a pairwise stable network. By way of contradiction, suppose that there does not exist $i, j \in N$ such that $g = g^N - ij$. Either g is the complete network g^N or a network obtained by deleting more than one link from g^N. If $g = g^N$, by Step 4, for some $ij \in L(g)$, g is defeated by $g - ij$. Otherwise, by Steps 2 and 3, for some $ij \notin L(g)$, g is defeated by $g + ij$. In both cases, g is not pairwise stable, a contradiction.

We now prove the "if" part. For some $i, j \in N$, let $g = g^N - ij$. Let $kl \in L(g)$. If $|N| > 3$, then $g - kl$ contains only one component. By Step 2, $g - kl$ is defeated by g. If $|N| = 3$, then $g - kl$ contains two components, and k and l belong to different components. By Step 3, $g - kl$ is defeated by g. In both cases, g is not defeated by $g - kl$. Since $g + ij = g^N$, by Step 4, g is not defeated by $g + ij$. Altogether, we conclude that g is pairwise stable.

Step 6: *For the value function v^α, any pairwise stable network is not link invariant.*

Proof. Let g be a pairwise stable network. By Step 5, for some $i, j \in N$, $g = g^N - ij$. Let $k \in N^c$ be a new agent and $g_0 = g \oplus k$. Since agents i and k belong to different components of g_0, by Step 3, g_0 is defeated by $g_1 = g_0 + ik$. Since g_1 is obtained by deleting more than one link from $g^{N\cup\{k\}}$ with $|C(g_1)| = 1$, by Step 2, g_1 is defeated by $g_2 = g_1 + ij$. By applying Step 2 repeatedly, we construct an improving path $\{g_2, \ldots, g_r\}$, where for each $t = 2, \ldots, r-1$, g_{t+1} is obtained by adding a link to g_t, and for some $l \in N$, $g_r = g^{N\cup\{k\}} - lk$. Then $\{g_0, g_1, g_2, \ldots, g_r\}$ is an improving path. By Step 5, g_r is pairwise stable and a limiting network of g_0. Since $ij \in L(g_2)$ and g_r is obtained by adding links to g_2, $ij \in L(g_r)$. Therefore, $L(g) \neq L(g_r) \cap L^N$ and thus g is not *link invariant.* □

Proof of Theorem 2: We prove (iii) first, and then (iv), (i), and (ii).

(iii) Let $g = (N, L) \in \mathcal{G}^N$ be a pairwise stable network and $k \in N^c$ be a new agent. By Proposition 2 in Jackson and Wolinsky (1996), if $c < \delta - \delta^2$, then the only pairwise stable network is g^N. Therefore, it suffices to show that

g^N is *distance invariant*. Let $g' \in \mathcal{G}^{N\cup\{k\}}$ be such that $g' \neq g^{N\cup\{k\}}$. For all $i,j \in N \cup \{k\}$ with $ij \notin L(g')$, $u_i(g'+ij) - u_i(g') \geq \delta - \delta^2 - c > 0$ and $u_j(g'+ij) - u_j(g') \geq \delta - \delta^2 - c > 0$, so that g' is defeated by $g'+ij$. Since this is true for all $g' \in \mathcal{G}^{N\cup\{k\}}$ with $g' \neq g^{N\cup\{k\}}$, the only limiting network from $g^N \oplus k$ is $g^{N\cup\{k\}}$. Therefore, for all $i,j \in N$, $d(i,j;g^N) = d(i,j;g^{N\cup\{k\}}) = 1$, which implies that g^N is *distance invariant.*

(iv) Let $g = (N,L) \in \mathcal{G}^N$ be a pairwise stable network, $k \in N^c$ be a new agent, and $g_0 = g \oplus k \in \mathcal{G}^{N\cup\{k\}}$. Since for all $i \in N$, $u_i(g_0) - u_i(g_0 + ik) = -\delta + c > 0$, g_0 is not defeated by $g_0 + ik$. For all $i,j \in N$ with $ij \notin L$, $u_i(g_0) - u_i(g_0+ij) = u_i(g) - u_i(g+ij)$ and $u_j(g_0) - u_j(g_0+ij) = u_j(g) - u_j(g+ij)$, which implies that g_0 is defeated by $g_0 + ij$ if and only if g is defeated by $g + ij$. Since g is pairwise stable, for all $i,j \in N$ with $ij \notin L$, g is not defeated by $g+ij$, or equivalently, g_0 is not defeated by $g_0 + ij$. Similarly, we can show that for all $i,j \in N$ with $ij \in L$, g_0 is not defeated by $g_0 - ij$. Since g_0 is not defeated by any adjacent network, it is pairwise stable, and thus g is *network invariant.*

(i) The proof is divided into three cases. Let $g = (N,L) \in \mathcal{G}^N$ be a pairwise stable network and $k \in N^c$ be a new agent.

Case 1: $c < \delta$. First, suppose that for some $ij \notin L$, $|C(g+ij)| < |C(g)|$. Since for i, the cost of link ij is c and the benefit of link ij is more than or equal to δ, $u_i(g+ij) - u_i(g) \geq \delta - c > 0$, which implies that $u_i(g+ij) > u_i(g)$. Similarly, $u_j(g+ij) > u_j(g)$. Therefore, g is defeated by $g+ij$, which implies that a pairwise stable network has only one component, that is, $|C(g)| = 1$.

Let $g_0 = g \oplus k$. Since $|C(g_0)| = 2$, g_1 defeats g_0 if and only if $g_1 = g_0 + hk$ for some $h \in N$. Let $\{g_0, g_1, g_2, \ldots, g_l\}$ be an improving path from g_0. Then, for $t = 1, \ldots, l-1$, $|C(g_t)| \geq |C(g_{t+1})|$. Since $|C(g_1)| = 1$, for all $t = 1, \ldots, l$, $|C(g_t)| = 1$. Since this is true for any improving path, g is *connectedness invariant.*

Case 2: $c = \delta$.

Step 1: *If* $|C(g+ij)| < |C(g)|$, *then* g *does not defeat* $g + ij$.

Proof. Let $h_i, h_j \in C(g)$ be the components of g containing i and j respectively. Let $h_i = (N_i, L_i)$, $h_j = (N_j, L_j)$, and $h = (N_i \cup N_j, L_i \cup L_j \cup \{ij\})$.

First, suppose that both h_i and h_j are singletons. Since for i, the cost of link ij is c and the benefit of link ij is δ, $u_i(g+ij)-u_i(g) = \delta - c = 0$. Similarly, $u_j(g+ij) - u_j(g) = 0$. Therefore, g does not defeat $g + ij$. Now suppose that at least one of h_i and h_j, say h_i, is not a singleton. Since h_i is not a singleton, $u_j(g+ij)-u_j(g) > \delta - c = 0$. Also, $u_i(g+ij)-u_i(g) \geq \delta - c = 0$. Altogether, $g + ij$ defeats g, which implies that g does not defeat $g + ij$.

Step 2: *If $g \in \mathcal{G}^N$ is pairwise stable, g is the empty network or $|C(g)| = 1$.*

Proof. Since the proof is obvious for $|N| \leq 2$, we assume that $|N| > 2$. Suppose by way of contradiction that there is a nonempty pairwise stable network g with $|C(g)| > 1$. Then, there is a non-singleton component $h_1 \in C(g)$ and another component $h_2 \in C(g)$. Let $i \in N(h_1)$ and $j \in N(h_2)$. By Step 1, $g + ij$ defeats g, which contradicts the pairwise stability of g.

Step 3: *A pairwise stable network is connectedness invariant.*

Proof. By Step 2, g is either the empty network or $|C(g)| = 1$. If g is the empty network, we can easily show that $g \oplus k$ is pairwise stable, which implies that g is *network invariant*. By Remark 1, g is *connectedness invariant*.

Now suppose that $|C(g)| = 1$. Since $|N| > 2$, g is a non-singleton. Let $g_0 = g \oplus k$. Since $|C(g_0)| = 2$, by the pairwise stability of g and Step 1, g_1 defeats g_0 if and only if $g_1 = g_0 + hk$ for some $h \in N$. Let $\{g_0, g_1, g_2, \ldots, g_l\}$ be an improving path from g_0. By Step 1, for $t = 1, \ldots, l-1$, $|C(g_{t+1})| \leq |C(g_t)|$. Since $|C(g_1)| = 1$, for all $t = 1, \ldots, l$, $|C(g_t)| = 1$. Since this is true for any improving path, g is *connectedness invariant.*

Case 3: $c > \delta$. By (iv), any pairwise stable network is *network invariant.* By Remark 1, it is *connectedness invariant.*

(ii) Suppose that g^N is pairwise stable. Let $k \in N^c$ be a new agent and $g_0 = g^N \oplus k \in \mathcal{G}^{N \cup \{k\}}$. Since the proof is obvious for $|N| \leq 2$, we assume that $|N| > 2$. Since g^N is pairwise stable, for all $i, j \in N$, $u_i(g^N) - u_i(g^N - ij) \geq 0$, which implies that $c \leq \delta - \delta^2$. If $c < \delta - \delta^2$, by (iii) and Remark 1, g^N is *link invariant.*

From now on, we assume that $c = \delta - \delta^2$. We will show that $\mathcal{L}(g_0) = \{g_0 + ik | i \in N\}$. For all $i \in N \cup \{k\}$, since $u_i(g_0 + ik) - u_i(g_0) = \delta - c > 0$, g_0 is defeated by $g_0 + ik$. Similarly, we can show that for all $j \in N$ with

$j \neq i$, $g_0 + ik$ is not defeated by $g_0 + ik + jk$ and for all $j, l \in N$, $g_0 + ik$ is not defeated by $g_0 + ik - jl$. Therefore, for all $i \in N$, $\{g_0, g_0 + ik\}$ is an improving path, and $g_0 + ik$ is pairwise stable.

Moreover, for all $i, j \in N$, since $u_i(g_0) - u_i(g_0 - ij) = \delta - \delta^2 - c = 0$, g_0 is not defeated by $g_0 - ij$. Hence, any improving path must be such that for some $i \in N$, $\{g_0, g_0 + ik\}$, which implies that $\mathcal{L}(g_0) = \{g_0 + ik | i \in N\}$. Since for all $g' \in \mathcal{L}(g_0)$, $L(g^N) = L(g') \cap L^N$, g^N is *link invariant.* $\square$

Proof of Theorem 3: Proposition 4 in Jackson and Wolinsky (1996) shows that (i) a pairwise stable network $g = (N, L) \in \mathcal{G}^N$ can be partitioned into fully intraconnected components, each of which has a different number of nodes, and that $m > n^2$, where m is the number of nodes in one component of g and n is the next largest in size. In the proof, it is also shown that (ii) if for some $i, j \in N$, $n_j \leq \max\{n_h | ih \in L\}$, then agent i strictly prefers to have a link with agent j. In addition, it can easily be shown that (iii) if for some $i \in N$, $n_i = 0$, then agent i wants to have a link with any other agents.

First, we prove the "if" part. Let $N \in \mathcal{N}$. Let g^N be the initial network and $k \in N^c$ be a new agent. Note that g^N is always pairwise stable. For $N' \subsetneq N$, let $L(k, N') = \{ik | i \in N'\}$ be the set of all links connecting k and any other agents in N', and $g(k, N') = (N \cup \{k\}, L^N \cup L(k, N'))$. By (ii) and (iii), for all $N' \subsetneq N$, g' defeats $g(k, N')$ if and only if $g' = g(k, N') + ij$ for some $i, j \in N \cup \{k\}$ with $ij \notin L(k, N')$. Therefore, an improving path from $g_0 = g^N \oplus k$ is of the form $\{g_0, \ldots, g_l\}$, where for all $t = 0, \ldots, l-1$, $g_{t+1} = g_i + ik$ for some $i \in N$. Hence, $\mathcal{L}(g_0) = \{g^{N \cup \{k\}}\}$, which implies that g^N is *link invariant.*

We now prove the "only if" part. Let $g \in \mathcal{G}^N$ be a pairwise stable network and $k \in N^c$ be a new agent. By way of contradiction, suppose that g is *link invariant*, but not complete. By (i) and (iii), this can happen only when $|N| > 4$, and moreover, there are at least two components of g. Let agent i be in the largest component of g and agent j be in the next largest component of g. Let $g_0 = g \oplus k$. By (ii) and (iii), agents k and i want to add link ik, so that g_0 is defeated by $g_0 + ik$. Also, in $g_0 + ik$, j and k want to add link jk, i.e., $g_0 + ik$ is defeated by $g_0 + ik + jk$. By applying this argument repeatedly, we construct an improving path $\{g_0, g_0 + ik, g_0 + ik + jk, \ldots, g^{N \cup \{k\}}\}$. Since

$L(g) \neq L(g^{N \cup \{k\}}) \cap L^N$, g is not *link invariant,* a contradiction. $\square$

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