Network Bads: How Bottlenecks Matter*

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Abstract

This paper examines a model where bads, such as water pollutants, infectious diseases, and liquidity shocks, arise at a source node and may be transmitted to a sink node as a flow through a network. Two players wish to decrease network bads, which is determined by the extent of bads at the source node and by the maximum transmission of bads through the network. One player can act to mitigate bads at the source node while the other player can act to reduce bads through the network. Each player incurs the cost of action but benefits from the decrease in network bads. The bottlenecks of the network affect the maximum transmission of bads and the behavior of the players. I characterize efficient profiles and equilibria in terms of the bottlenecks. Interestingly, the player who acts in an equilibrium may not be the player who must act in the efficient profile. Overall, strategic inaction leads to inefficiency. Unless there is an efficient equilibrium where neither player acts, no efficient profile is an equilibrium and no equilibrium is efficient. I study cooperative solutions where the players jointly choose an efficient profile and make transfers to each other. Such cooperative solutions always exist.

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1 Introduction

When there is transmission of bads through a network, the structure of the network affects the behavior of economic players. Players may want to control the source of bads or the bottlenecks of the network. The structure of the network determines the bottlenecks, and in turn, the bottlenecks determine the transmission of bads through the network. Therefore, by controlling the bottlenecks, players can control the transmission of bads. However, players behave strategically and equilibrium outcomes may not be efficient.

For example, when water pollutants are carried through rivers and pipelines, public agencies can view the whole system of rivers and pipelines as a network and manage the bottlenecks of the network. They can also manage the source of pollution. If public agencies manage the source and the network separately, they may not take action that leads to efficient outcomes.[1]

Cooperative approaches through transfers between players can help achieve efficient outcomes. Sharing human resources and monetary funds can be examples of transfers. During the outbreak of Severe Acute Respiratory Syndrome (SARS) in 2003, by sharing human resources, public health agencies worked together to prevent the disease. Also, in the ongoing debt crisis, European governments are seeking cooperative agreements including the creation and expansion of a bailout fund, the European Financial Stability Facility (EFSF), to resolve the crisis.

This paper examines a model where bads arise at a source node and may be transmitted to a sink node through a network. Two players, player 1 and player 2, wish to decrease the extent of bads transmitted to the sink node. Player 1 can act to mitigate the Extent of Bads at the Source (EBS) while player 2 can act to reduce the Maximum Transmission of Bads (MTB) through the network. The minimum of the mitigated EBS and the reduced MTB is assumed to determine the extent of bads transmitted to the sink node, which is called the extent of network bads. Each player incurs the cost of action but benefits from the decrease in the extent of network bads. Each player's action to decrease the extent of network bads can be viewed as a public good.

I characterize efficient strategy profiles and equilibria in terms of the bottlenecks of the network and the extent of bads at the source, as well as costs and benefits. In an efficient profile the joint payoff is maximized. In an equilibrium each player maximizes his own payoff given the other player's strategy.

In every efficient profile and in every equilibrium, either (a) player 1 acts to mitigate the EBS or (b) player 2 acts to reduce the MTB through the network or (c) neither player acts. Because the minimum of the mitigated EBS and the reduced MTB determines the extent of network bads, it is neither an efficient profile nor an equilibrium for both players to act. Thus, one player may free-ride on the other player's action. In case (a), player 2 free-rides on player 1's action. In case (b), player 1 free-rides on player 2's action.

[1] When infectious diseases are transmitted through social networks between individuals, public agencies may want to control social networks to reduce the transmission of diseases. For studies on disease control, see Glass et al. [8] and Viboud et al. [22]. When liquidity shocks are contagious through financial networks between banks, liquid banks may want to control financial networks to reduce the contagion of shocks. For studies on financial contagion, see Allen and Gale [1] and Leitner [17].

If either player 1 or player 2 acts, in an efficient profile, the marginal cost of action equals to the marginal joint loss from network bads, whereas in an equilibrium, the marginal cost equals to the marginal private loss. Thus, in case (a) and in case (b), the equilibrium level of action is less than the efficient level.

For further comparison, my analysis is divided into three cases depending on the bottleneck capacity. In a network a bottleneck is an arc used to its capacity when the maximum transmission occurs. The total capacity of bottlenecks, or the bottleneck capacity, determines the maximum transmission of bads through the network. Whether the bottleneck capacity is small, intermediate, or large depends on the extent of bads at the source and the efficient level of action.

If the bottleneck capacity is small, the Maximum Transmission of Bads (MTB) is less than the Extent of Bads at the Source (EBS) mitigated to the efficient level. Thus, even if player 1 mitigates the EBS to the efficient level, the extent of network bads does not decrease. Similarly, when he mitigates to the equilibrium level, the extent does not decrease. Therefore, in every efficient profile and in every equilibrium, player 1 does not act at all. Either (b) player 2 acts or (c) neither player acts.

If the bottleneck capacity is large, the EBS is less than or equal to the MTB reduced to the efficient level. Thus, even if player 2 reduces the MTB to the efficient level, the extent of network bads does not decrease. Therefore, in every efficient profile, player 2 does not act at all. Either (a) player 1 acts or (c) neither player acts. However, in equilibrium, player 1 always acts. That is, there is no equilibrium where player 2 acts or neither player acts.

If the bottleneck capacity is intermediate, the MTB is greater than or equal to the EBS mitigated to the efficient level, and the EBS is greater than the MTB reduced to the efficient level. Thus, both players can decrease the extent of network bads to the efficient level. In every efficient profile, either (a) player 1 acts or (b) player 2 acts. Costs and benefits determine who takes action in an efficient profile. However, in equilibrium, both players may take no action or wrong action while they try to free-ride on each other's action. Specifically, there may be an inefficient equilibrium where neither player acts. More interestingly, the player who acts in an equilibrium may not be the player who must act in the efficient profile. That is, the efficient player may not act in equilibrium. Overall, strategic inaction leads to inefficiency.[2]

Unless there is an efficient equilibrium where neither player acts, no efficient profile is an equilibrium and no equilibrium is efficient. Thus, the players may not have incentives to act for efficiency. To provide incentives for efficiency, I introduce cooperative solutions, where the players jointly choose an efficient profile and make transfers to each other. Each player is better off by using a cooperative solution than by maximizing his payoff alone. I show that such cooperative solutions always exist. Therefore, both players can always achieve efficiency by using cooperative solutions. If the bottleneck capacity is intermediate, in a cooperative solution, the efficient player acts to decrease network bads and gets a positive net transfer from the other.

[2]In a best-shot model of public goods free-riding behavior is observed both theoretically and experimentally. See Harrison and Hirshleifer [12] and Hirshleifer [13] for reference. In a special case, where the EBS equals to the MTB, my model can be viewed as a best-shot model of public goods in networks.

The contribution of this paper is threefold.

First, this paper contributes to the literature on public goods in networks by considering public bads transmitted through networks. Public goods, and also public bads, are transmitted through various networks. Knowledge and information can be examples of public goods.[3] Infectious diseases and liquidity shocks can be examples of public bads. In the literature there are two different approaches: The network interaction approach assumes that networks are given and fixed; the network formation approach assumes that networks are chosen by players. Bramoullé and Kranton [3] is an example of the former; Galeotti and Goyal [7] is an example of the latter.[4] A third approach is explored in this paper: A network is given but not fixed. Some player may have limited discretion to control the network. For example, suppose that a disease breaks out in one area connected through a transportation network. This disease can be transmitted through the network to another area. An agency may want to reduce the transmission of the disease by controlling the network. The agency's efforts to reduce the transmission can be viewed as public goods in the network.

Second, this paper contributes to the literature on adversarial networks by introducing non-adversarial players. While these players are affected by network bads, they have different tools to deal with the bads. Thus, it is important to coordinate the players' behavior to achieve better outcomes. In the literature, however, it is assumed that players are adversarial. Washburn and Wood [23] study a zero-sum game between an evader and an interdictor in a given network. The evader chooses a path to move through the network while the interdictor chooses an arc in the network to stop the evader. Baccara and Bar-Isaac [2] study a game between a law enforcement agency and a criminal organization. The agency allocates its resources to detect criminals while the criminals form an organization to carry out illegal activities. In this paper, players are non-adversarial. Players coordinate with each other to decrease network bads. Cooperative solutions are studied to help the coordination.

Third, this paper contributes to the literature on flow games by introducing strategic behavior. Kalai and Zemel [15] introduce a coalitional game, called a flow game, where the worth of a coalition is defined as the value of a maximum flow through the network restricted to the members of the coalition. Every flow game is totally balanced and thus has a non-empty core. A minimum cut is used to find a core allocation.[5] In this paper, players make strategic choices to decrease the extent of network bads, which is constrained by the value of a maximum flow through a network. Equilibrium strategies, as well as efficient strategies, are constructed on the concept of a minimum cut.

The rest of this paper is organized as follows. Section 2 develops a strategic network model. Section 3 analyzes efficient profiles and equilibria and introduces cooperative solutions. Section 4 concludes.

[3]Conley and Udry [5] show how farmers learn agricultural practices from other farmers.

[4]Cho [4] also studies a network model for public goods. For a review of the literature, see Goyal [9] and Jackson [14].

[5]For studies on flow games, see Granot and Granot [10], Kalai and Zemel [16], Potters et al. [19], and Reijnierse et al. [20]. Also, see Granot and Maschler [11] and Van den Nouweland et al. [21] for related studies on spanning network games.

2 Model

Let $N := \{1, 2, \ldots, n\}$ be the set of nodes with $n \geq 2$. Node 1 is called the source and node n is called the sink. Let $A \subseteq \{(i, j) \in N \times N : i \neq j\}$ be the set of arcs, where each arc is an ordered pair of distinct nodes. Each arc (i, j) is directed from node i to node j. The Extent of Bads at the Source (EBS) is given as a number $x > 0$. These bads are transmitted through an arc from one node to another node. For each $(i, j) \in A$, the maximum transmission of bads through arc (i, j) from node i to node j is described by a number $y_{ij} > 0$, which is called the capacity of arc (i, j).[6] Let $y := (y_{ij})_{(i,j)\in A}$ be the vector of arc capacities. The collection (N, A, y) is called a *network*.

Two players, player 1 and player 2, wish to decrease the extent of bads transmitted to the sink. Both players choose their strategies simultaneously and independently.

Player 1 can act to mitigate the extent of bads at the source. Player 1 chooses a *mitigation strategy* m with $0 \leq m \leq x$. The set of mitigation strategies for player 1 is denoted by M. By choosing $m \in M$, player 1 mitigates the extent of bads at the source to $x - m$.

Player 2 can act to reduce the maximum transmission of bads through network (N, A, y). Player 2 chooses a *reduction strategy* $r := (r_{ij})_{(i,j)\in A}$ with $0 \leq r_{ij} \leq y_{ij}$ for each $(i, j) \in A$. The set of reduction strategies for player 2 is denoted by R. By choosing $r = (r_{ij})_{(i,j)\in A} \in R$, player 2 reduces the maximum transmission of bads through arc (i, j) to $y_{ij} - r_{ij}$. The total reduction is defined as $\sum_{(i,j)\in A} r_{ij}$.

For each $(m, r) \in M \times R$, the extent of bads transmitted to the sink is called the extent of *network bads* and denoted by $e(m, r)$. How to define $e(m, r)$ will be explained later.

Both players lose $l(e(m, r))$ from $e(m, r)$, where $l(\cdot)$ is a *loss function* with $l(0) = 0$. In other words, both players benefit from the decrease in the extent of network bads. However, each player incurs the cost of action. Player 1 pays $c_1(m)$ for $m \in M$, where $c_1(\cdot)$ is a *mitigation cost function* with $c_1(0) = 0$. Player 2 pays $c_2\left(\sum_{(i,j)\in A} r_{ij}\right)$ for $r = (r_{ij})_{(i,j)\in A} \in R$, where $\sum_{(i,j)\in A} r_{ij}$ is the total reduction, and $c_2(\cdot)$ is a *reduction cost function* with $c_2(0) = 0$. Assume that $l(\cdot)$, $c_1(\cdot)$, and $c_2(\cdot)$ are twice continuously differentiable, strictly increasing, and strictly convex. That is, $l'(\cdot) > 0$, $l''(\cdot) > 0$, $c_1'(\cdot) > 0$, $c_1''(\cdot) > 0$, $c_2'(\cdot) > 0$, and $c_2''(\cdot) > 0$. Both players earn a constant worth w.

For each $(m, r) \in M \times R$, the payoff of player 1 is

$$u_1(m, r) = w - c_1(m) - l(e(m, r))$$

and the payoff of player 2 is

$$u_2(m, r) = w - c_2\left(\sum_{(i,j)\in A} r_{ij}\right) - l(e(m, r)).$$

Here the transmission of bads creates a public bad. Both mitigation strategy and reduction strategy are public goods to decrease the public bad.

[6] This capacity can be viewed as a plausible upper bound for the transmission of bads through arc (i, j). For a discussion of plausible upper bounds in risk analysis, see Paté-Cornell [18].

Let us define $e(m, r)$ the extent of network bads. The transmission of bads from the source to the sink is modeled as a flow through a network. Formally, for each $r \in R$, a flow through network $(N, A, y - r)$ is a vector $f := (f_{ij})_{(i,j)\in A}$ satisfying the following constraints:

$$0 \leq f_{ij} \leq y_{ij} - r_{ij} \qquad \text{for each } (i, j) \in A; \tag{1}$$

$$f_{i1} = 0 \qquad \text{for each } (i, 1) \in A; \tag{2}$$

$$f_{ni} = 0 \qquad \text{for each } (n, i) \in A; \tag{3}$$

$$\sum_{j:(j,i)\in A} f_{ji} = \sum_{j:(i,j)\in A} f_{ij} \qquad \text{for each } i \in N \setminus \{1, n\}. \tag{4}$$

Constraint (1) says that each arc flow f_{ij} is non-negative and constrained by the reduced arc capacity $y_{ij} - r_{ij}$. Constraint (2) says that there is no arc flow to the source. Constraint (3) says that there is no arc flow from the sink. Constraint (4) says that at each node i, except for the source and the sink, the total arc flow to node i equals to the total arc flow from node i. That is, there is no change in the total arc flow at node $i \neq 1, n$. For each $r \in R$, let $F(r)$ be the set of all flows through network $(N, A, y - r)$.

For each $r \in R$, the value of a maximum flow through network $(N, A, y - r)$ is defined as

$$\begin{aligned} f(r) \quad = \quad & \max \sum_{(i,n)\in A} f_{in} \\ & \text{subject to } f \in F(r). \end{aligned} \tag{5}$$

That is, $f(r)$ is the maximum total arc flow to the sink through network $(N, A, y - r)$. The value of a maximum flow through network $(N, A, y - r)$ is also called the Maximum Transmission of Bads (MTB).

For each $(m, r) \in M \times R$, the extent of network bads is defined as

$$e(m, r) = \min\{x - m, f(r)\}.$$

That is, the extent of network bads is defined as the minimum of the mitigated EBS and the reduced MTB. For each $(m, r) \in M \times R$, if $x - m \leq f(r)$, the extent of network bads is $e(m, r) = x - m$. If $f(r) \leq x - m$, however, $e(m, r) = f(r)$. Now consider an example where pollution may be carried through a river.

Example 1 A river is described as a network (N, A, y), where $N = \{1, 2, 3\}$, $A = \{(1, 2), (2, 3)\}$, and $y = (y_{12}, y_{23})$. Pollution arises at node 1. The extent of pollution is measured by a number x. The capacity y_{ij} of arc (i, j) shows the maximum amount of pollution carried through (i, j). Players want to decrease pollution carried to node 3. Player 1 chooses m to mitigate pollution at the source. Player 2 chooses $r = (r_{12}, r_{23})$ to reduce pollution carried through the river. See Figure 1 for an illustration. □

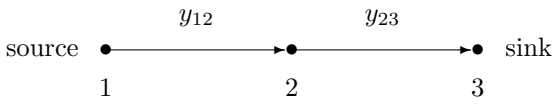


Figure 1 River Pollution

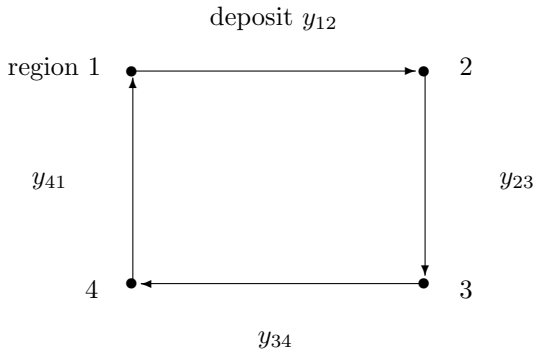


Figure 2 Financial Contagion

Consider another example where a liquidity shock may be transmitted through an interbank deposit market. This example is motivated by Allen and Gale [1].

Example 2 An interbank deposit market is described as a network (N, A, y), where $N = \{1, 2, 3, 4\}$, $A = \{(1, 2), (2, 3), (3, 4), (4, 1)\}$, and $y = (y_{12}, y_{23}, y_{34}, y_{41})$. Each node $i \in N$ represents a region with a group of banks. Each arc $(i, j) \in A$ indicates the possibility of banks in region i holding deposits in region j. The total amount of such deposits is denoted by y_{ij} and called the capacity of arc (i, j). A liquidity shock arises in region 1. The extent x of the shock is measured by the amount of liquidity that banks in region 1 demand. To meet liquidity demands, banks in region 1 withdraw deposits in region 2, and similarly for banks in other regions. Thus, the capacity y_{ij} of arc (i, j) constrains the transmission of the shock from region i to region j. See Figure 2. Players want to decrease the transmission of the shock to region 4. Player 1 chooses m to bail out banks in region 1. Player 2 chooses $r = (r_{12}, r_{23}, r_{34}, r_{41})$ to regulate the total amount of withdrawals. Because the liquidity shock is transmitted from region 1 to region 4 through a path in the network, the maximum transmission of the shock can be written as

$$f(r) = \min_{(i,j)\in P} y_{ij} - r_{ij},$$

where $P = \{(1, 2), (2, 3), (3, 4)\}$ is the path from the source to the sink.[7] $\square$

In the above examples a bottleneck can be defined as an arc with the smallest capacity along a path in a network. Player 2 may act to reduce the capacity of a bottleneck to decrease the transmission of bads. In general, however, there may be more than one bottleneck. Let us define bottlenecks formally.

A cut $C := (C_1, C_n)$ is an ordered partition of the node set N with $1 \in C_1$ and $n \in C_n$. For each cut $C = (C_1, C_n)$, an arc $(i, j) \in A$ is a cut arc of C if $i \in C_1$ and $j \in C_n$. For each cut $C = (C_1, C_n)$, let $A(C) := \{(i, j) \in A : i \in C_1 \text{ and } j \in C_n\}$ be the set of all cut arcs of C.

[7] This example can be extended to consider a *downward spiral effect*. Imagine a situation where banks in region 4 withdraw deposits in region 1 when they get a shock from region 3. These withdrawals will result in a new liquidity shock to region 1. A repeated game-theoretic model can be used to study this situation.

For each $r \in R$, a cut C^r is a *minimum cut* in network $(N, A, y - r)$ if for each cut C,

$$\sum_{(i,j)\in A(C^r)} y_{ij} - r_{ij} \leq \sum_{(i,j)\in A(C)} y_{ij} - r_{ij}. \tag{6}$$

The left hand side of inequality (6) is called the capacity of a minimum cut in network $(N, A, y - r)$.

Bottlenecks are cut arcs with minimum total capacity. For each $r \in R$, if C^r is a minimum cut in network $(N, A, y - r)$, then $A(C^r)$ is the set of all cut arcs of C^r. Each cut arc $(i, j) \in A(C^r)$ is called a *bottleneck* of network $(N, A, y - r)$. The capacity of a minimum cut is also called the *bottleneck capacity*.

This bottleneck capacity determines the Maximum Transmission of Bads (MTB). The *max-flow min-cut theorem* says that for each $r \in R$, the value of a maximum flow through network $(N, A, y - r)$ equals to the capacity of a minimum cut in the network.[8] Formally, for each $r \in R$,

$$f(r) = \sum_{(i,j)\in A(C^r)} y_{ij} - r_{ij}. \tag{7}$$

In other words, the MTB through network $(N, A, y - r)$ equals to the bottleneck capacity of the network.

Player 2 may act to reduce the bottleneck capacity to decrease the MTB. A reduction strategy $r \in R$ is a *bottleneck reduction strategy* if there is a minimum cut C in network (N, A, y) such that for each $(i, j) \in A(C)$, $r_{ij} > 0$, and for each $(i, j) \notin A(C)$, $r_{ij} = 0$. Note that each cut arc $(i, j) \in A(C)$ is a bottleneck of network (N, A, y). Player 2 decreases the MTB by reducing the bottleneck capacity.

If player 2 chooses a bottleneck reduction strategy, the total reduction is equal to the decrease in the MTB. The vector of zeros $\underline{0}$ is called the *zero reduction strategy*. The MTB through network (N, A, y) is denoted by $f(\underline{0})$. For each cut C, the MTB through network (N, A, y) is less than or equal to the total capacity of cut arcs of C. Formally, for each cut C,

$$f(\underline{0}) \leq \sum_{(i,j)\in A(C)} y_{ij}. \tag{8}$$

From the max-flow min-cut theorem (7) and inequality (8), for each $r \in R$, we can show that

$$\sum_{(i,j)\in A} r_{ij} \geq f(\underline{0}) - f(r). \tag{9}$$

That is, the total reduction is greater than or equal to the decrease in the MTB.

From the max-flow min-cut theorem (7), if $r \in R$ is a bottleneck reduction strategy, we can show that

$$\sum_{(i,j)\in A} r_{ij} = f(\underline{0}) - f(r). \tag{10}$$

That is, if player 2 chooses a bottleneck reduction strategy, the total reduction is equal to the decrease in the MTB. Consider the following example.

[8] Ford and Fulkerson [6] introduce the maximum flow problem (5) and show the max-flow min-cut theorem.

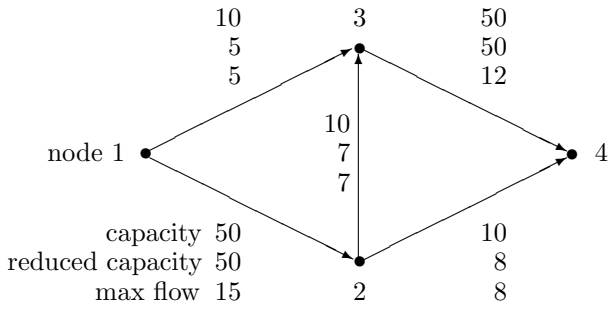


Figure 3 Bottleneck Reduction

Example 3 A network is given as (N, A, y), where $N = \{1, 2, 3, 4\}$, $A = \{(1, 2), (1, 3), (2, 3), (2, 4), (3, 4)\}$, and $y = (y_{12}, y_{13}, y_{23}, y_{24}, y_{34}) = (50, 10, 10, 10, 50)$. There is a unique minimum cut $C = (\{1, 2\}, \{3, 4\})$ in network (N, A, y). The set of all cut arcs of C is $A(C) = \{(1, 3), (2, 3), (2, 4)\}$. These arcs are the bottlenecks of this network. Suppose that player 2 chooses a bottleneck reduction strategy $r = (0, 5, 3, 2, 0)$. Player 2 does not reduce the capacities of arcs $(1, 2)$ and $(3, 4)$, neither of which is a bottleneck. See Figure 3. Because $f(\underline{0}) = 30$ and $f(r) = 20$, the total reduction is equal to the decrease in the MTB, as in equality (10). □

3 Analysis

This section analyzes efficient (strategy) profiles and equilibria in my model.

A strategy $\tilde{m} \in M$ is a *best response* of player 1 to $r \in R$ if for each $m \in M$, $u_1(\tilde{m}, r) \geq u_1(m, r)$. For each $r \in R$, let $BR_1(r) \subseteq M$ be the set of player 1's best responses to r. A strategy $\tilde{r} \in R$ is a *best response* of player 2 to $m \in M$ if for each $r \in R$, $u_2(m, \tilde{r}) \geq u_2(m, r)$. For each $m \in M$, let $BR_2(m) \subseteq R$ be the set of player 2's best responses to m. A strategy profile $(\tilde{m}, \tilde{r}) \in M \times R$ is an *equilibrium* if $\tilde{m}$ is a best response to $\tilde{r}$ and $\tilde{r}$ is a best response to $\tilde{m}$.

For each $(m, r) \in M \times R$, the joint payoff is $U(m, r) = 2w - c_1(m) - c_2\left(\sum_{(i,j)\in A} r_{ij}\right) - 2l(e(m, r))$. A strategy profile $(\tilde{m}, \tilde{r}) \in M \times R$ is *efficient* if for each $(m, r) \in M \times R$, $U(\tilde{m}, \tilde{r}) \geq U(m, r)$. That is, in an efficient (strategy) profile, the joint payoff is maximized.

To keep my analysis simple I make two assumptions on the loss and cost functions. Recall that x is the Extent of Bads at the Source (EBS) and $f(\underline{0})$ is the Maximum Transmission of Bads (MTB) through network (N, A, y).

(A1) The marginal private loss at x is greater than the marginal mitigation cost at 0, i.e., $l'(x) > c_1'(0)$. The marginal joint loss at 0 is less than the marginal mitigation cost at x, i.e., $2l'(0) < c_1'(x)$.

(A2) The marginal private loss at $f(\underline{0})$ is greater than the marginal reduction cost at 0, i.e., $l'(f(\underline{0})) > c_2'(0)$. The marginal joint loss at 0 is less than the marginal reduction cost at $f(\underline{0})$, i.e., $2l'(0) < c_2'(f(\underline{0}))$.

By analyzing efficient profiles and equilibria I want to answer the following questions: How does network (N, A, y) affect efficient profiles and equilibria? How much mitigation and reduction do players choose in each efficient profile and in each equilibrium? Do players have incentives for efficiency?

3.1 Efficient Profiles

In an efficient profile there is at most one player who acts to decrease the extent of network bads.

Remark 1 In every efficient profile, either (a) player 1 acts to mitigate the EBS or (b) player 2 acts to reduce the MTB or (c) neither player acts. For each $(m, r) \in M \times R$, if both players act, that is, if $m \neq 0$ and $r \neq \underline{0}$, then (m, r) is not efficient. Because the extent of network bads is defined as the minimum of the mitigated EBS and the reduced MTB, one player's action is not necessary to decrease the extent of network bads. Without this unnecessary action, the joint payoff increases. □

In an efficient profile where player 2 acts, the total reduction is equal to the decrease in the MTB.

Remark 2 In an efficient profile, if player 2 acts to reduce the MTB, the total reduction is equal to the decrease in the MTB. For each $(0, r)$ with $f(r) \leq x$, if the total reduction is greater than the decrease in the MTB, that is, if $\sum_{(i,j)\in A} r_{ij} > f(\underline{0}) - f(r)$, then $(0, r)$ is not efficient. With a bottleneck reduction strategy $\tilde{r}$ such that $f(\tilde{r}) = f(r)$, the joint payoff increases. □

When player 1 acts to mitigate the EBS, the efficient level of mitigation $\bar{m}$ is determined by the condition

$$c_1'(\bar{m}) = 2l'(x - \bar{m}).$$

That is, the marginal mitigation cost at $\bar{m}$ equals to the marginal joint loss at $x - \bar{m}$. Note that $0 < \bar{m} < x$. For each $(m, \underline{0})$ with $x - m \leq f(\underline{0})$, the joint payoff can be written as $\bar{U}_1(m) = 2w - c_1(m) - 2l(x - m)$, which is maximized at $m = \bar{m}$. The existence of a unique value for $\bar{m}$ is guaranteed by (A1).

When player 2 acts to reduce the MTB, in an efficient profile, the total reduction is equal to the decrease in the MTB. Let q denote a level of total reduction. The efficient level of reduction $\bar{q}$ is determined by the condition

$$c_2'(\bar{q}) = 2l'(f(\underline{0}) - \bar{q}).$$

That is, the marginal reduction cost at $\bar{q}$ equals to the marginal joint loss at $f(\underline{0}) - \bar{q}$. Note that $0 < \bar{q} < f(\underline{0})$. For each $(0, r)$ with $f(r) \leq x$, the joint payoff can be written as $\bar{U}_2(q) = 2w - c_2(q) - 2l(f(\underline{0}) - q)$, because $q = f(\underline{0}) - f(r)$. This joint payoff is maximized at $q = \bar{q}$. The existence of a unique value for $\bar{q}$ is guaranteed by (A2).

Let $\bar{R} := \{\bar{r} \in R : \sum_{(i,j)\in A} \bar{r}_{ij} = \bar{q} \text{ and } \bar{q} = f(\underline{0}) - f(\bar{r})\}$ be a set of reduction strategies with total reduction $\bar{q}$.

My analysis is divided into three cases depending on the bottleneck capacity of network (N, A, y), which equals to the MTB through the network. The bottleneck capacity of network (N, A, y) is *small* if $f(\underline{0}) < x - \bar{m}$, *intermediate* if $x - \bar{m} \leq f(\underline{0})$ and $f(\bar{r}) < x$, and *large* if $x \leq f(\bar{r})$.

If the bottleneck capacity is small, that is, if $f(\underline{0}) < x - \bar{m}$, the MTB through network (N, A, y) is less than the EBS mitigated to the efficient level. Thus, even if player 1 acts to mitigate the EBS to the efficient level $x - \bar{m}$, the extent of network bads does not decrease. Therefore, in every efficient profile, player 1 does not act at all. Whether player 2 acts depends on the loss and cost functions. The following proposition summarizes this result. All proofs are presented in Appendix A.

Proposition 1 If the bottleneck capacity is small, that is, if $f(\underline{0}) < x - \bar{m}$, every efficient profile is $(0, \underline{0})$ or $(0, \bar{r})$. Furthermore, if $2l(f(\underline{0})) < c_2\left(\sum \bar{r}_{ij}\right) + 2l(f(\bar{r}))$, then $(0, \underline{0})$ is the only efficient profile. However, if $2l(f(\underline{0})) > c_2\left(\sum \bar{r}_{ij}\right) + 2l(f(\bar{r}))$, then $(0, \bar{r})$ is the only efficient profile.

If the bottleneck capacity is intermediate, that is, if $x - \bar{m} \leq f(\underline{0})$ and $f(\bar{r}) < x$, the MTB through network (N, A, y) is greater than or equal to the EBS mitigated to the efficient level, and the EBS is greater than the MTB through network $(N, A, y - \bar{r})$. Thus, both players can decrease the extent of network bads to the efficient level. Therefore, in every efficient profile, either player 1 or player 2 acts to decrease the extent of network bads. There is no efficient profile in which neither player acts. Whether player 1 acts or player 2 acts depends on the loss and cost functions.

Proposition 2 If the bottleneck capacity is intermediate, that is, if $x - \bar{m} \leq f(\underline{0})$ and $f(\bar{r}) < x$, every efficient profile is $(0, \bar{r})$ or $(\bar{m}, \underline{0})$. Furthermore, if $c_2\left(\sum \bar{r}_{ij}\right) + 2l(f(\bar{r})) < c_1(\bar{m}) + 2l(x - \bar{m})$, then $(0, \bar{r})$ is the only efficient profile. However, if $c_2\left(\sum \bar{r}_{ij}\right) + 2l(f(\bar{r})) > c_1(\bar{m}) + 2l(x - \bar{m})$, then $(\bar{m}, \underline{0})$ is the only efficient profile.

If $(\bar{m}, \underline{0})$ is the only efficient profile, player 1 is called the efficient player. However, if $(0, \bar{r})$ is the only efficient profile, player 2 is called the efficient player.

Example 4 Consider network (N, A, y) in Example 3. The MTB through network (N, A, y) is $f(\underline{0}) = 30$. Suppose that $x = 36$. Assume that $l(e(m, r)) = (e(m, r))^2$, $c_1(m) = m^2$, and $c_2\left(\sum r_{ij}\right) = \left(\sum r_{ij}\right)^2$. Because $4(x - \bar{m}) = 2\bar{m}$ and $x = 36$, we have $\bar{m} = 24$. Because $4(f(\underline{0}) - \bar{q}) = 2\bar{q}$ and $f(\underline{0}) = 30$, we have $\bar{q} = 20$. Suppose that player 2 chooses $\bar{r} = (0, 10, 5, 5, 0)$. The MTB through network $(N, A, y - \bar{r})$ is $f(\bar{r}) = 10$. Because $x - \bar{m} = 12 \leq 30 = f(\underline{0})$ and $f(\bar{r}) = 10 < 36 = x$, the bottleneck capacity is intermediate. Because $c_2\left(\sum \bar{r}_{ij}\right) + 2l(f(\bar{r})) = 600 < 864 = c_1(\bar{m}) + 2l(x - \bar{m})$, from Proposition 2, $(0, \bar{r})$ is efficient. □

In the example above, player 2 is the efficient player. However, in the example below, player 1 is the efficient player.

Example 5 Consider Example 4 but now assume that $c_2\left(\sum r_{ij}\right) = 4\left(\sum r_{ij}\right)^2$. Because $4(f(\underline{0})-\bar{q}) = 8\bar{q}$ and $f(\underline{0}) = 30$, we have $\bar{q} = 10$. Suppose that player 2 chooses $\bar{r} = (0, 4, 3, 3, 0)$. The MTB through $(N, A, y - \bar{r})$ is $f(\bar{r}) = 20$. Because $x - \bar{m} = 12 \leq 30 = f(\underline{0})$ and $f(\bar{r}) = 20 < 36 = x$, the bottleneck capacity is intermediate. Because $c_2\left(\sum \bar{r}_{ij}\right) + 2l(f(\bar{r})) = 1200 > 864 = c_1(\bar{m}) + 2l(x - \bar{m})$, from Proposition 2, $(\bar{m}, \underline{0})$ is efficient. □

If the bottleneck capacity is large, that is, if $x \leq f(\bar{r})$, the EBS is less than or equal to the MTB through network $(N, A, y - \bar{r})$. Thus, even if player 2 acts to reduce the MTB to the efficient level $f(\bar{r})$, the extent of network bads does not decrease. Therefore, in every efficient profile, player 2 does not act at all. Whether player 1 acts depends on the loss and cost functions.

Proposition 3 If the bottleneck capacity is large, that is, if $x \leq f(\bar{r})$, every efficient profile is $(0, \underline{0})$ or $(\bar{m}, \underline{0})$. Furthermore, if $2l(x) < c_1(\bar{m}) + 2l(x - \bar{m})$, then $(0, \underline{0})$ is the only efficient profile. However, if $2l(x) > c_1(\bar{m}) + 2l(x - \bar{m})$, then $(\bar{m}, \underline{0})$ is the only efficient profile.

To sum up, in every efficient profile, either (a) player 1 acts to mitigate the EBS or (b) player 2 acts to reduce the MTB through the network or (c) neither player acts. In case (a), player 2 free-rides on player 1's mitigation effort. The efficient level of mitigation is $\bar{m}$, where the marginal mitigation cost equals to the marginal joint loss. In case (b), player 1 free-rides on player 2's reduction effort. The efficient level of reduction is $\sum_{(i,j)\in A} \bar{r}_{ij} = \bar{q}$, where the marginal reduction cost equals to the marginal joint loss.

3.2 Equilibria

When player 1 mitigates the EBS, in an equilibrium, the level of mitigation $\hat{m}$ is determined by the condition

$$c_1'(\hat{m}) = l'(x - \hat{m}).$$

That is, the marginal mitigation cost at $\hat{m}$ equals to the marginal private loss at $x - \hat{m}$. Note that $0 < \hat{m} < x$ and $\hat{m} < \bar{m}$. For each (m, r) with $x - m \leq f(r)$, player 1's payoff is $\hat{u}_1(m) = w - c_1(m) - l(x - m)$, which is maximized at $m = \hat{m}$. The existence of a unique value for $\hat{m}$ is guaranteed by (A1).

Given $r \in R$, the MTB through the reduced network $(N, A, y - r)$ is $f(r)$, and player 1's best response depends on $f(r)$. Player 1's best response to $r \in R$ is $m = 0$ or $m = \hat{m}$ or both.

Lemma 1 The set of player 1's best responses to $r \in R$ is

$$BR_1(r) = \begin{cases} \{0\} & \text{if } f(r) < x - \hat{m}; \\ \{0\} & \text{if } x - \hat{m} \leq f(r) < x \text{ and } l(f(r)) < c_1(\hat{m}) + l(x - \hat{m}); \\ \{0, \hat{m}\} & \text{if } x - \hat{m} \leq f(r) < x \text{ and } l(f(r)) = c_1(\hat{m}) + l(x - \hat{m}); \\ \{\hat{m}\} & \text{if } x - \hat{m} \leq f(r) < x \text{ and } l(f(r)) > c_1(\hat{m}) + l(x - \hat{m}); \\ \{\hat{m}\} & \text{if } x \leq f(r). \end{cases}$$

The following is a necessary condition for player 2's best responses, which is analogous to Remark 2.

Remark 3 If $r \in R$ is a best response to $m \in M$ such that $f(r) \leq x - m$, then $\sum_{(i,j)\in A} r_{ij} = f(\underline{0}) - f(r)$. That is, the total reduction is equal to the decrease in the MTB. For each (m, r) with $f(r) \leq x - m$, if $\sum_{(i,j)\in A} r_{ij} > f(\underline{0}) - f(r)$, then r is not a best response to m. With a bottleneck reduction strategy $\tilde{r}$ such that $f(\tilde{r}) = f(r)$, player 2's payoff increases. □

When player 2 reduces the MTB, in an equilibrium, the total reduction is equal to the decrease in the MTB. The equilibrium level of reduction $\hat{q}$ is determined by the condition

$$c_2'(\hat{q}) = l'(f(\underline{0}) - \hat{q}).$$

That is, the marginal reduction cost at $\hat{q}$ equals to the marginal private loss at $f(\underline{0})-\hat{q}$. Note that $0 < \hat{q} < f(\underline{0})$ and $\hat{q} < \bar{q}$. For each (m, r) with $f(r) \leq x - m$, player 2's payoff is $\hat{u}_2(q) = w - c_2(q) - l(f(\underline{0}) - q)$, where q denotes a level of total reduction. This payoff is maximized at $q = \hat{q}$. The existence of a unique value for $\hat{q}$ is guaranteed by (A2).

Let $\hat{R} := \{\hat{r} \in R : \sum_{(i,j)\in A} \hat{r}_{ij} = \hat{q} \text{ and } \hat{q} = f(\underline{0}) - f(\hat{r})\}$ be a set of reduction strategies with total reduction $\hat{q}$.

Given $m \in M$, the mitigated EBS is $x - m$, and player 2's best response depends on $x - m$. Player 2's best response to $m \in M$ is $r = \underline{0}$ or $r = \hat{r}$ or both.

Lemma 2 The set of player 2's best responses to $m \in M$ is

$$BR_2(m) = \begin{cases} \{\underline{0}\} & \text{if } x - m \leq f(\hat{r}); \\ \{\underline{0}\} & \text{if } f(\hat{r}) < x - m \text{ and } l(x - m) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r})); \\ \{\underline{0}\} \cup \hat{R} & \text{if } f(\hat{r}) < x - m \text{ and } l(x - m) = c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r})); \\ \hat{R} & \text{if } f(\hat{r}) < x - m \text{ and } l(x - m) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r})). \end{cases}$$

From Lemma 1, player 1's best response is $m = 0$ or $m = \hat{m}$. From Lemma 2, player 2's best response is $r = \underline{0}$ or $r = \hat{r}$. There are four strategy profiles $(0, \underline{0})$, $(0, \hat{r})$, $(\hat{m}, \underline{0})$, and $(\hat{m}, \hat{r})$ for equilibria.

At a glance we can see that $(\hat{m}, \hat{r})$ is not an equilibrium. If $(\hat{m}, \hat{r})$ is an equilibrium, $\hat{m}$ is a best response to $\hat{r}$. From Lemma 1, if $\hat{m}$ is a best response to $\hat{r}$, it is necessary that $x - \hat{m} \leq f(\hat{r})$. From Lemma 2, if $x - \hat{m} \leq f(\hat{r})$, then $\hat{r}$ is not a best response to $\hat{m}$. This is a contradiction.

Now there remain three strategy profiles $(0, \underline{0})$, $(0, \hat{r})$, and $(\hat{m}, \underline{0})$ for equilibria. In these strategy profiles there is at most one player who acts to decrease the extent of network bads. In strategy profile $(0, \underline{0})$, neither player acts. In strategy profile $(0, \hat{r})$, player 2 acts to reduce the MTB through network (N, A, y) by $\sum_{(i,j)\in A} \hat{r}_{ij} = \hat{q}$. In strategy profile $(\hat{m}, \underline{0})$, player 1 acts to mitigate the EBS by $\hat{m}$.

I want to know when these strategy profiles are equilibria. From Lemma 1, player 1's best response to $r \in R$ depends on $f(r)$. From Lemma 2, player 2's best response is $r = \underline{0}$ or $r = \hat{r}$. There are six cases for equilibria. Table 1 shows these cases.

$r = \underline{0}$	$r = \hat{r}$	Results
$f(\underline{0}) < x - \hat{m}$	$f(\hat{r}) < x - \hat{m}$	Proposition 4
	$x - \hat{m} \leq f(\hat{r}) < x$	Contradiction
	$x \leq f(\hat{r})$	Contradiction
$x - \hat{m} \leq f(\underline{0}) < x$	$f(\hat{r}) < x - \hat{m}$	Proposition 5
	$x - \hat{m} \leq f(\hat{r}) < x$	Proposition 6
	$x \leq f(\hat{r})$	Contradiction
$x \leq f(\underline{0})$	$f(\hat{r}) < x - \hat{m}$	Proposition 7
	$x - \hat{m} \leq f(\hat{r}) < x$	Proposition 8
	$x \leq f(\hat{r})$	Proposition 9

Table 1 Six Cases for Equilibria

In Table 1 there are three contradictions. Because $f(\hat{r}) < f(\underline{0})$, if $f(\underline{0}) < x - \hat{m}$, then $f(\hat{r}) < x - \hat{m}$, which contradicts $x - \hat{m} \leq f(\hat{r})$ and $x \leq f(\hat{r})$. Similarly, if $f(\underline{0}) < x$, then $f(\hat{r}) < x$, which contradicts $x \leq f(\hat{r})$. Now I characterize equilibria in each of the six cases.

Proposition 4 Suppose that $f(\underline{0}) < x - \hat{m}$.

(i) If $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, then $(0, \underline{0})$ is the only equilibrium.

(ii) If $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, then $(0, \hat{r})$ is the only equilibrium.

Because $f(\underline{0}) < x - \hat{m}$, from Lemma 1, $\hat{m}$ is not a best response to $\underline{0}$. Thus, $(\hat{m}, \underline{0})$ is not an equilibrium. In Proposition 4, every equilibrium is $(0, \underline{0})$ or $(0, \hat{r})$.

Proposition 5 Suppose that $x - \hat{m} \leq f(\underline{0}) < x$ and $f(\hat{r}) < x - \hat{m}$.

(i) If $l(f(\underline{0})) < c_1(\hat{m}) + l(x - \hat{m})$ and $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, then $(0, \underline{0})$ is the only equilibrium.

(ii) If $l(f(\underline{0})) < c_1(\hat{m}) + l(x - \hat{m})$ and $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, then $(0, \hat{r})$ is the only equilibrium.

(iii) If $l(f(\underline{0})) > c_1(\hat{m}) + l(x - \hat{m})$ and $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, then $(\hat{m}, \underline{0})$ is the only equilibrium.

(iv) If $l(f(\underline{0})) > c_1(\hat{m}) + l(x - \hat{m})$ and $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r})) > l(x - \hat{m})$, then $(0, \hat{r})$ and $(\hat{m}, \underline{0})$ are the only pure strategy equilibria.

(v) If $l(f(\underline{0})) > c_1(\hat{m}) + l(x - \hat{m})$ and $l(x - \hat{m}) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, then $(0, \hat{r})$ is the only equilibrium.

If $x-\hat{m} \leq f(\underline{0}) < x$ and $l(f(\underline{0})) < c_1(\hat{m}) + l(x-\hat{m})$, from Lemma 1, $\hat{m}$ is not a best response to $\underline{0}$. Thus, $(\hat{m}, \underline{0})$ is not an equilibrium. Therefore, in (i) and (ii) of Proposition 5, every equilibrium is $(0, \underline{0})$ or $(0, \hat{r})$.

However, if $x-\hat{m} \leq f(\underline{0}) < x$ and $l(f(\underline{0})) > c_1(\hat{m}) + l(x-\hat{m})$, from Lemma 1, 0 is not a best response to $\underline{0}$. Thus, $(0, \underline{0})$ is not an equilibrium. Therefore, in (iii) through (v) of Proposition 5, every equilibrium is $(0, \hat{r})$ or $(\hat{m}, \underline{0})$.

Example 6 Recall Example 4. Since $2(x-\hat{m}) = 2\hat{m}$ and $x = 36$, we have $\hat{m} = 18$. Since $2(f(\underline{0}) - \hat{q}) = 2\hat{q}$ and $f(\underline{0}) = 30$, we have $\hat{q} = 15$. If player 2 chooses $\hat{r} = (0, 5, 5, 5, 0)$, the MTB through $(N, A, y - \hat{r})$ is $f(\hat{r}) = 15$. From (iv) of Proposition 5, both $(0, \hat{r})$ and $(\hat{m}, \underline{0})$ are equilibria. □

In the example above, there are two equilibria $(0, \hat{r})$ and $(\hat{m}, \underline{0})$. In Example 4, however, $(0, \bar{r})$ is efficient. Thus, the efficient player, player 2, may not act in equilibrium. If player 2 does not act by choosing $\underline{0}$, knowing that player 1's best response to $\underline{0}$ is $\hat{m}$, player 2 can free-ride on player 1's action in $(\hat{m}, \underline{0})$. Even if player 2 acts by choosing $\hat{r}$, the equilibrium level of action is less than the efficient level. That is, $\sum \hat{r}_{ij} < \sum \bar{r}_{ij}$.

Proposition 6 Suppose that $x - \hat{m} \leq f(\hat{r})$ and $f(\underline{0}) < x$.

(i) If $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$ and $l(f(\underline{0})) < c_1(\hat{m}) + l(x-\hat{m})$, then $(0, \underline{0})$ is the only equilibrium.

(ii) If $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$ and $l(f(\underline{0})) > c_1(\hat{m}) + l(x-\hat{m})$, then $(\hat{m}, \underline{0})$ is the only equilibrium.

(iii) If $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$ and $l(f(\underline{0})) < c_1(\hat{m}) + l(x-\hat{m})$, then $(0, \hat{r})$ is the only equilibrium.

(iv) If $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$ and $l(f(\underline{0})) > c_1(\hat{m}) + l(x-\hat{m}) > l(f(\hat{r}))$, then $(0, \hat{r})$ and $(\hat{m}, \underline{0})$ are the only pure strategy equilibria.

(v) If $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$ and $l(f(\hat{r})) > c_1(\hat{m}) + l(x-\hat{m})$, then $(\hat{m}, \underline{0})$ is the only equilibrium.

If $f(\underline{0}) < x$ and $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, from Lemma 2, $\hat{r}$ is not a best response to 0. Thus, $(0, \hat{r})$ is not an equilibrium. In (i) and (ii) of Proposition 6, every equilibrium is $(0, \underline{0})$ or $(\hat{m}, \underline{0})$.

If $f(\underline{0}) < x$ and $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, from Lemma 2, $\underline{0}$ is not a best response to 0. Thus, $(0, \underline{0})$ is not an equilibrium. In (iii) through (v) of Proposition 6, every equilibrium is $(0, \hat{r})$ or $(\hat{m}, \underline{0})$.

Example 7 Recall Example 5. Since $2(x-\hat{m}) = 2\hat{m}$ and $x = 36$, we have $\hat{m} = 18$. Since $2(f(\underline{0}) - \hat{q}) = 8\hat{q}$ and $f(\underline{0}) = 30$, we have $\hat{q} = 6$. If player 2 chooses $\hat{r} = (0, 2, 2, 2, 0)$, the MTB through $(N, A, y - \hat{r})$ is $f(\hat{r}) = 24$. From (iv) of Proposition 6, both $(0, \hat{r})$ and $(\hat{m}, \underline{0})$ are equilibria. □

In the example above, there are two equilibria $(0, \hat{r})$ and $(\hat{m}, \underline{0})$. In Example 5, however, $(\bar{m}, \underline{0})$ is efficient. Thus, the efficient player, player 1, may not act in equilibrium. If player 1 does not act by choosing 0, knowing

that player 2's best response to 0 is $\hat{r}$, player 1 can free-ride on player 2's action in $(0, \hat{r})$. Even if player 1 acts by choosing $\hat{m}$, the equilibrium level of action is less than the efficient level. That is, $\hat{m} < \bar{m}$.

Proposition 7 Suppose that $x \leq f(\underline{0})$ and $f(\hat{r}) < x - \hat{m}$.

(i) If $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, then $(\hat{m}, \underline{0})$ is the only equilibrium.

(ii) If $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r})) > l(x - \hat{m})$, then $(0, \hat{r})$ and $(\hat{m}, \underline{0})$ are the only pure strategy equilibria.

(iii) If $l(x - \hat{m}) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, then $(0, \hat{r})$ is the only equilibrium.

Proposition 8 Suppose that $x \leq f(\underline{0})$ and $x - \hat{m} \leq f(\hat{r}) < x$.

(i) If $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, then $(\hat{m}, \underline{0})$ is the only equilibrium.

(ii) If $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$ and $l(f(\hat{r})) > c_1(\hat{m}) + l(x - \hat{m})$, then $(\hat{m}, \underline{0})$ is the only equilibrium.

(iii) If $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$ and $l(f(\hat{r})) < c_1(\hat{m}) + l(x - \hat{m})$, then $(0, \hat{r})$ and $(\hat{m}, \underline{0})$ are the only pure strategy equilibria.

If $x \leq f(\underline{0})$, from Lemma 1, 0 is not a best response to $\underline{0}$. Thus, $(0, \underline{0})$ is not an equilibrium. Therefore, in Propositions 7 and 8, every equilibrium is $(0, \hat{r})$ or $(\hat{m}, \underline{0})$.

Proposition 9 If $x \leq f(\hat{r})$, then $(\hat{m}, \underline{0})$ is the only equilibrium.

If $x \leq f(\hat{r})$, from Lemmas 1 and 2, $\hat{m}$ is the only best response to $\underline{0}$ and $\hat{r}$, and $\underline{0}$ is the only best response to $\hat{m}$. Thus, in Proposition 9, $(\hat{m}, \underline{0})$ is the only equilibrium.

To sum up, in every equilibrium, either (a) player 1 acts to mitigate the EBS or (b) player 2 acts to reduce the MTB through the network or (c) neither player acts. In case (a), player 2 free-rides on player 1's mitigation effort. The equilibrium level of mitigation is $\hat{m}$, where the marginal mitigation cost equals to the marginal private loss. In case (b), player 1 free-rides on player 2's reduction effort. The equilibrium level of reduction is $\sum_{(i,j)\in A} \hat{r}_{ij} = \hat{q}$, where the marginal reduction cost equals to the marginal private loss.

Now I compare equilibria with efficient profiles. Table 2 summarizes the results.

If either (a) player 1 acts or (b) player 2 acts, in an efficient profile, the marginal cost of action equals to the marginal joint loss. In an equilibrium, the marginal cost of action equals to the marginal private loss. Thus, in case (a), the equilibrium level of mitigation is less than the efficient level, that is, $\hat{m} < \bar{m}$. Also, in case (b), the equilibrium level of reduction is less than the efficient level, that is, $\sum \hat{r}_{ij} < \sum \bar{r}_{ij}$.

Bottleneck capacity	Profiles	(a) Player 1 acts	(b) Player 2 acts	(c) Neither acts
Small	EF	No EF	$(0, \bar{r})$	$(0, \underline{0})$
	EQ	No EQ	$(0, \hat{r})$	$(0, \underline{0})$
Intermediate	EF	$(\bar{m}, \underline{0})$	$(0, \bar{r})$	No EF
	EQ	$(\hat{m}, \underline{0})$	$(0, \hat{r})$	$(0, \underline{0})$
Large	EF	$(\bar{m}, \underline{0})$	No EF	$(0, \underline{0})$
	EQ	$(\hat{m}, \underline{0})$	No EQ	No EQ

Table 2 Efficient Profiles (EF) versus Equilibria (EQ)

From Proposition 1, if the bottleneck capacity of network (N, A, y) is small, that is, if $f(\underline{0}) < x - \bar{m}$, every efficient profile is $(0, \underline{0})$ or $(0, \bar{r})$. From Proposition 4, if $f(\underline{0}) < x - \hat{m}$, every equilibrium is $(0, \underline{0})$ or $(0, \hat{r})$. Because $f(\underline{0}) < x - \bar{m}$ implies $f(\underline{0}) < x - \hat{m}$, if the bottleneck capacity of network (N, A, y) is small, every equilibrium is $(0, \underline{0})$ or $(0, \hat{r})$. Thus, in every efficient profile and in every equilibrium, either (b) player 2 acts or (c) neither player acts. There may be an efficient equilibrium $(0, \underline{0})$ in which neither player acts.

From Proposition 2, if the bottleneck capacity of network (N, A, y) is intermediate, that is, if $x - \bar{m} \leq f(\underline{0})$ and $f(\bar{r}) < x$, every efficient profile is $(0, \bar{r})$ or $(\bar{m}, \underline{0})$. Thus, in every efficient profile, either (a) player 1 acts or (b) player 2 acts. However, no efficient profile is an equilibrium and no equilibrium is efficient. There may be an equilibrium $(0, \underline{0})$ in which neither player acts. Further, as shown in Examples 6 and 7, the player who acts in an equilibrium may not be the player who must act in the efficient profile. That is, the efficient player may not act in equilibrium.

From Proposition 3, if the bottleneck capacity of network (N, A, y) is large, that is, if $x \leq f(\bar{r})$, every efficient profile is $(0, \underline{0})$ or $(\bar{m}, \underline{0})$. Thus, in every efficient profile, either (a) player 1 acts or (c) neither player acts. From Proposition 9, if $x \leq f(\hat{r})$, then $(\hat{m}, \underline{0})$ is the only equilibrium. Because $x \leq f(\bar{r})$ implies $x \leq f(\hat{r})$, if the bottleneck capacity of network (N, A, y) is large, $(\hat{m}, \underline{0})$ is the only equilibrium. Thus, in equilibrium, player 1 always acts. However, the equilibrium level of action is less than the efficient level.

Unless $(0, \underline{0})$ is an efficient equilibrium, no efficient profile is an equilibrium and no equilibrium is efficient. The exception is observed only if the bottleneck capacity is small. Therefore, except for this case, we can conclude that players 1 and 2 do not have incentives to choose efficient profiles.

3.3 Cooperative Solutions

In my model players may not have incentives to choose efficient profiles. Now transfers are allowed between players. I want to know if it is possible to make transfers that provide incentives to choose efficient profiles. To answer this question I introduce a coalitional game and study cooperative solutions for the game.

A *coalition* $S \subseteq \{1,2\}$ is a non-empty set of players. A *coalitional game* v is a function that associates with each coalition S a number $v(S)$, which is called the value of coalition S.

In my model a coalitional game v is defined as follows:

$$\begin{aligned} v(\{1\}) &= \max_{m \in M} u_1(m, \underline{0}); \\ v(\{2\}) &= \max_{r \in R} u_2(0, r); \\ v(\{1,2\}) &= \max_{(m,r) \in M \times R} U(m, r). \end{aligned}$$

That is, each player maximizes his payoff without any action from the other if he belongs to a singleton coalition. Players 1 and 2 maximize the joint payoff if they belong to the same coalition.

Player 1 maximizes $u_1(m, \underline{0})$ by choosing a best response to $\underline{0}$. From Lemma 1, player 1's best response is 0 or $\hat{m}$. If 0 is a best response to $\underline{0}$, then $v(\{1\}) = u_1(0, \underline{0})$. If $\hat{m}$ is a best response to $\underline{0}$, then $v(\{1\}) = u_1(\hat{m}, \underline{0})$.

Player 2 maximizes $u_2(0, r)$ by choosing a best response to 0. From Lemma 2, player 2's best response is $\underline{0}$ or $\hat{r}$. If $\underline{0}$ is a best response to 0, then $v(\{2\}) = u_2(0, \underline{0})$. If $\hat{r}$ is a best response to 0, then $v(\{2\}) = u_2(0, \hat{r})$.

Players 1 and 2 maximize the joint payoff $U(m, r)$ by choosing an efficient profile.

The coalitional game v is *superadditive* if $v(\{1\}) + v(\{2\}) \leq v(\{1,2\})$. An *allocation* (a_1, a_2) is a vector of payoffs with transfers, where a_i is the allocation of player $i = 1, 2$. An allocation (a_1, a_2) is in the set of *imputations* of the coalitional game v if $a_1 \geq v(\{1\})$, $a_2 \geq v(\{2\})$, and $a_1 + a_2 = v(\{1,2\})$. If the coalitional game v is superadditive, the set of imputations of v is non-empty.

Proposition 10 The coalitional game v is superadditive.

If (m, r) is an efficient profile and (a_1, a_2) is an allocation in the set of imputations of v, players 1 and 2 jointly choose (m, r) and make transfers in a way that player $i = 1, 2$ gets a net transfer of $a_i - u_i(m, r)$. Each player is better off by using this cooperative solution than by maximizing his payoff alone. Therefore, it is possible to make transfers that provide incentives for efficiency.

Remark 4 If the bottleneck capacity is intermediate, in a cooperative solution, the efficient player acts to decrease the extent of network bads and gets a positive net transfer from the other. If player 1 is the efficient player, he gets at least $v(\{1\}) - u_1(\bar{m}, \underline{0})$ from player 2. However, if player 2 is the efficient player, he gets at least $v(\{2\}) - u_2(0, \bar{r})$ from player 1.

For every coalitional game with two players, superadditivity is equivalent to convexity. Thus, the coalitional game v is convex. Because every convex game is totally balanced and every totally balanced game is a flow game, the coalitional game v is a flow game. Kalai and Zemel [15] show how to construct a flow game.

4 Conclusion

This paper introduces a model where bads are transmitted from a source to a sink as a flow through a network. Two players wish to decrease network bads. One player can act to mitigate the extent of bads at the source. The other player can act to reduce the maximum transmission of bads through the network. Each player incurs the cost of action but benefits from the decrease in network bads.

I characterize efficient profiles and equilibria in this model. The equilibrium level of action is less than the efficient level. Interestingly, the player who acts in an equilibrium may not be the player who must act in the efficient profile. Further, there may an equilibrium where neither player acts. However, this equilibrium may not be efficient. Therefore, strategic inaction may lead to inefficiency.

Unless there is an efficient equilibrium where neither player acts, no efficient profile is an equilibrium and no equilibrium is efficient. Thus, the players may not have incentives to choose efficient profiles. I also study cooperative solutions where the players jointly choose an efficient profile and make transfers to each other. The players can achieve efficiency by using cooperative solutions.

An extension of this model is to consider multiple network players. Networks may be controlled by various players. Coordination between players is a key to reduce the transmission of bads through networks. Another extension is to study the transmission of network goods. Online data streaming can be an example. Content providers, such as Netflix and YouTube, and network service providers, such as AT&T and Verizon, have different interests that may lead to inefficient use of networks. In these extensions cooperative solutions can help achieve social efficiency.

Appendix A

Propositions 1 through 3 characterize efficient profiles in my model. From Remark 1, if $m \neq 0$ and $r \neq \underline{0}$, then (m, r) is not efficient. Thus, in the following proofs, we will only consider (m, r) with $m = 0$ or $r = \underline{0}$. Also, from Remark 2, if $f(r) \leq x$ and if $\sum_{(i,j)\in A} r_{ij} > f(\underline{0}) - f(r)$, then $(0, r)$ is not efficient. Henceforth, if $f(r) \leq x$, we will only consider $(0, r)$ with $\sum_{(i,j)\in A} r_{ij} = f(\underline{0}) - f(r)$.

Proof of Proposition 1. Suppose that $f(\underline{0}) < x - \bar{m}$. First, let $(0, r)$ be a profile in which player 1 does not act at all. Because $f(\underline{0}) < x - \bar{m}$ implies $f(r) < x$, we have $e(0, r) = f(r)$ and $\sum_{(i,j)\in A} r_{ij} = f(\underline{0}) - f(r)$. The joint payoff is $U(0, r) = 2w - c_2\left(f(\underline{0}) - f(r)\right) - 2l(f(r))$, which can be written as $\bar{U}_2(q) = 2w - c_2\left(q\right) - 2l(f(\underline{0}) - q)$, where $q = f(\underline{0}) - f(r)$. The function $\bar{U}_2(q)$ is maximized at $q = \bar{q}$. Because $\bar{q} = f(\underline{0}) - f(\bar{r})$, the joint payoff $U(0, r)$ is maximized at $r = \bar{r}$. The maximum joint payoff is $U(0, \bar{r}) = 2w - c_2\left(\sum \bar{r}_{ij}\right) - 2l(f(\bar{r}))$.

Second, let $(m, \underline{0})$ be a profile in which player 2 does not act at all. Because $f(\underline{0}) < x - \bar{m}$, the joint payoff $U(m, \underline{0})$ is maximized at $m = 0$. The maximum joint payoff is $U(0, \underline{0}) = 2w - 2l(f(\underline{0}))$.

Thus, if the bottleneck capacity is small, that is, if $f(\underline{0}) < x - \bar{m}$, every efficient profile is $(0, \underline{0})$ or $(0, \bar{r})$. Further, if $2l(f(\underline{0})) < c_2\left(\sum \bar{r}_{ij}\right) + 2l(f(\bar{r}))$, then $U(0, \underline{0}) > U(0, \bar{r})$, and $(0, \underline{0})$ is the only efficient profile.

However, if $2l(f(\underline{0})) > c_2\left(\sum \bar{r}_{ij}\right) + 2l(f(\bar{r}))$, then $U(0,\underline{0}) < U(0,\bar{r})$, and $(0,\bar{r})$ is the only efficient profile. □

Proof of Proposition 2. Suppose that $x - \bar{m} \leq f(\underline{0})$ and $f(\bar{r}) < x$. First, let $(0,r)$ be a profile in which player 1 does not act at all. If $x \leq f(r)$, then $e(0,r) = x$ and $U(0,r) = 2w - c_2\left(\sum_{(i,j)\in A} r_{ij}\right) - 2l(x)$. This joint payoff $U(0,r)$ is maximized at $r = \underline{0}$. Thus, $U(0,\underline{0}) = 2w - 2l(x)$. However, if $f(r) \leq x$, then $e(0,r) = f(r)$ and $\sum_{(i,j)\in A} r_{ij} = f(\underline{0}) - f(r)$. The joint payoff $U(0,r)$ can be written as $\bar{U}_2(q) = 2w - c_2(q) - 2l(f(\underline{0}) - q)$, where $q = f(\underline{0}) - f(r)$. The function $\bar{U}_2(q)$ is maximized at $q = \bar{q}$. Because $\bar{q} = f(\underline{0}) - f(\bar{r})$, the joint payoff $U(0,r)$ is maximized at $r = \bar{r}$. Thus, $U(0,\bar{r}) = 2w - c_2\left(\sum \bar{r}_{ij}\right) - 2l(f(\bar{r}))$.

Second, let $(m,\underline{0})$ be a profile in which player 2 does not act at all. If $f(\underline{0}) \leq x - m$, then $e(m,\underline{0}) = f(\underline{0})$ and $U(m,\underline{0}) = 2w - c_1(m) - 2l(f(\underline{0}))$. This joint payoff $U(m,\underline{0})$ is maximized at $m = 0$. Thus, $U(0,\underline{0}) = 2w - 2l(f(\underline{0}))$. However, if $x - m \leq f(\underline{0})$, then $e(m,\underline{0}) = x - m$ and $U(m,\underline{0}) = 2w - c_1(m) - 2l(x-m)$. The joint payoff $U(m,\underline{0})$ is maximized at $m = \bar{m}$. Thus, $U(\bar{m},\underline{0}) = 2w - c_1(\bar{m}) - 2l(x - \bar{m})$.

Now compare the joint payoffs for profiles $(0,\underline{0})$, $(0,\bar{r})$, and $(\bar{m},\underline{0})$. If $x \leq f(\underline{0})$, then $U(0,\underline{0}) = 2w - 2l(x)$ and $U(\bar{m},\underline{0}) = 2w - c_1(\bar{m}) - 2l(x - \bar{m})$. By the assumption (A1), $U(0,\underline{0}) < U(\bar{m},\underline{0})$. If $f(\underline{0}) \leq x$, then $U(0,\underline{0}) = 2w - 2l(f(\underline{0}))$ and $U(0,\bar{r}) = 2w - c_2\left(\sum \bar{r}_{ij}\right) - 2l(f(\bar{r}))$. By the assumption (A2), $U(0,\underline{0}) < U(0,\bar{r})$. In both cases, $(0,\underline{0})$ is not efficient.

Therefore, if the bottleneck capacity is intermediate, that is, if $x - \bar{m} \leq f(\underline{0})$ and $f(\bar{r}) < x$, every efficient profile is $(0,\bar{r})$ or $(\bar{m},\underline{0})$. Further, if $c_2\left(\sum \bar{r}_{ij}\right) + 2l(f(\bar{r})) < c_1(\bar{m}) + 2l(x - \bar{m})$, then $U(0,\bar{r}) > U(\bar{m},\underline{0})$, and $(0,\bar{r})$ is the only efficient profile. However, if $c_2\left(\sum \bar{r}_{ij}\right) + 2l(f(\bar{r})) > c_1(\bar{m}) + 2l(x - \bar{m})$, then $U(0,\bar{r}) < U(\bar{m},\underline{0})$, and $(\bar{m},\underline{0})$ is the only efficient profile. □

Proof of Proposition 3. Suppose that $x \leq f(\bar{r})$. First, let $(0,r)$ be a profile in which player 1 does not act at all. Because $x \leq f(\bar{r})$, the joint payoff $U(0,r)$ is maximized at $r = \underline{0}$. The maximum joint payoff is $U(0,\underline{0}) = 2w - 2l(x)$.

Second, let $(m,\underline{0})$ be a profile in which player 2 does not act at all. Because $x \leq f(\bar{r})$ implies $x - m < f(\underline{0})$, we have $e(m,\underline{0}) = x - m$ and $U(m,\underline{0}) = 2w - c_1(m) - 2l(x-m)$. The joint payoff $U(m,\underline{0})$ is maximized at $m = \bar{m}$. The maximum joint payoff is $U(\bar{m},\underline{0}) = 2w - c_1(\bar{m}) - 2l(x - \bar{m})$.

Thus, if the bottleneck capacity is large, that is, if $x \leq f(\bar{r})$, every efficient profile is $(0,\underline{0})$ or $(\bar{m},\underline{0})$. Further, if $2l(x) < c_1(\bar{m}) + 2l(x - \bar{m})$, then $U(0,\underline{0}) > U(\bar{m},\underline{0})$, and $(0,\underline{0})$ is the only efficient profile. However, if $2l(x) > c_1(\bar{m}) + 2l(x - \bar{m})$, then $U(0,\underline{0}) < U(\bar{m},\underline{0})$, and $(\bar{m},\underline{0})$ is the only efficient profile. □

Lemma 1 characterizes player 1's best responses to a reduction strategy r.

Proof of Lemma 1. Let $r \in R$ be any reduction strategy. Divide into three cases.

Case 1. Suppose that $f(r) < x - \hat{m}$. For each $m \in M$, if $x - m \leq f(r)$, then $e(m,r) = x - m$ and $u_1(m,r) = w - c_1(m) - l(x-m)$. Because $x - m \leq f(r)$ and $f(r) < x - \hat{m}$, the payoff is maximized at $m = x - f(r)$. Thus, $u_1(x - f(r), r) = w - c_1(x - f(r)) - l(f(r))$. For each $m \in M$, if $f(r) \leq x - m$, then $e(m,r) = f(r)$ and $u_1(m,r) = w - c_1(m) - l(f(r))$. Because $c_1(\cdot)$ is strictly increasing, the payoff is

maximized at $m = 0$. Thus, $u_1(0, r) = w - l(f(r))$. Because $x - f(r) > \hat{m}$ and $\hat{m} > 0$, $x - f(r) > 0$. Because $c_1(\cdot)$ is strictly increasing, $u_1(0, r) > u_1(x - f(r), r)$. Thus, $m = 0$ is the only best response.

Case 2. Suppose that $x - \hat{m} \leq f(r) < x$. For each $m \in M$, if $x - m \leq f(r)$, then $e(m, r) = x - m$ and $u_1(m, r) = w - c_1(m) - l(x - m)$. Because $x - m \leq f(r)$ and $x - \hat{m} \leq f(r)$, the payoff is maximized at $m = \hat{m}$. Thus, $u_1(\hat{m}, r) = w - c_1(\hat{m}) - l(x - \hat{m})$. For each $m \in M$, if $f(r) \leq x - m$, then $e(m, r) = f(r)$ and $u_1(m, r) = w - c_1(m) - l(f(r))$. Because $c_1(\cdot)$ is strictly increasing, the payoff is maximized at $m = 0$. Thus, $u_1(0, r) = w - l(f(r))$.

If $l(f(r)) < c_1(\hat{m}) + l(x - \hat{m})$, then $u_1(0, r) > u_1(\hat{m}, r)$. Thus, $m = 0$ is the only best response. If $l(f(r)) = c_1(\hat{m}) + l(x - \hat{m})$, then $u_1(0, r) = u_1(\hat{m}, r)$. Thus, both $m = 0$ and $m = \hat{m}$ are the best responses. If $l(f(r)) > c_1(\hat{m}) + l(x - \hat{m})$, then $u_1(\hat{m}, r) > u_1(0, r)$. Thus, $m = \hat{m}$ is the only best response.

Case 3. Suppose that $x \leq f(r)$. Because $x \leq f(r)$, for each $m \in M$, we have $x - m \leq f(r)$. Thus, $e(m, r) = x - m$ and $u_1(m, r) = w - c_1(m) - l(x - m)$. The payoff is maximized at $m = \hat{m}$. Thus, $m = \hat{m}$ is the only best response. □

Lemma 2 characterizes player 2's best responses to a mitigation strategy m.

From Remark 3, if $f(r) \leq x - m$ and if $\sum_{(i,j)\in A} r_{ij} > f(\underline{0}) - f(r)$, then r is not a best response to m. In the following proof, if $f(r) \leq x - m$, we will only consider a strategy r with $\sum_{(i,j)\in A} r_{ij} = f(\underline{0}) - f(r)$.

Proof of Lemma 2. Let $m \in M$ be any mitigation strategy. Divide into two cases.

Case 1. Suppose that $x - m \leq f(\hat{r})$. For each $r \in R$, if $x - m \leq f(r)$, then $e(m, r) = x - m$ and $u_2(m, r) = w - c_2\left(\sum_{(i,j)\in A} r_{ij}\right) - l(x - m)$. Because $c_2(\cdot)$ is strictly increasing, the payoff is maximized at $r = \underline{0}$. Thus, $u_2(m, \underline{0}) = w - l(x - m)$. For each $r \in R$, if $f(r) \leq x - m$, then $e(m, r) = f(r)$ and $\sum_{(i,j)\in A} r_{ij} = f(\underline{0}) - f(r)$. Thus, $u_2(m, r) = w - c_2(f(\underline{0}) - f(r)) - l(f(r))$. Because $f(r) \leq x - m$ and $x - m \leq f(\hat{r})$, the payoff is maximized when $f(r) = x - m$. For such a strategy r, we have $u_2(m, r) = w - c_2(f(\underline{0}) - x + m) - l(x - m)$. Because $x - m \leq f(\hat{r})$ and $f(\hat{r}) < f(\underline{0})$, we have $f(\underline{0}) - x + m > 0$. Because $c_2(\cdot)$ is strictly increasing, $u_2(m, \underline{0}) > u_2(m, r)$. Thus, $\underline{0}$ is the only best response.

Case 2. Suppose that $f(\hat{r}) < x - m$. For each $r \in R$, if $x - m \leq f(r)$, then $e(m, r) = x - m$ and $u_2(m, r) = w - c_2\left(\sum_{(i,j)\in A} r_{ij}\right) - l(x - m)$. Because $c_2(\cdot)$ is strictly increasing, the payoff is maximized at $r = \underline{0}$. Thus, $u_2(m, \underline{0}) = w - l(x - m)$. For each $r \in R$, if $f(r) \leq x - m$, then $e(m, r) = f(r)$ and $\sum_{(i,j)\in A} r_{ij} = f(\underline{0}) - f(r)$. Thus, $u_2(m, r) = w - c_2(f(\underline{0}) - f(r)) - l(f(r))$. This payoff can be written as $\hat{u}_2(q) = w - c_2(q) - l(f(\underline{0}) - q)$, where $q = f(\underline{0}) - f(r)$. The function $\hat{u}_2(q)$ is maximized at $q = \hat{q}$. Because $\hat{q} = f(\underline{0}) - f(\hat{r})$, the payoff $u_2(m, r)$ is maximized at $r = \hat{r}$. Thus, $u_2(m, \hat{r}) = w - c_2\left(\sum \hat{r}_{ij}\right) - l(f(\hat{r}))$.

If $l(x - m) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, then $u_2(m, \underline{0}) > u_2(m, \hat{r})$. Thus, $\underline{0}$ is the only best response. If $l(x - m) = c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, then $u_2(m, \underline{0}) = u_2(m, \hat{r})$. Thus, $\underline{0}$ and $\hat{r}$ are the best responses. If $l(x - m) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, then $u_2(m, \hat{r}) > u_2(m, \underline{0})$. Thus, $\hat{r}$ is the only best response. □

From Lemma 1, player 1's best responses are as follows:

$$BR_1(\underline{0}) = \begin{cases} \{0\} & \text{if } f(\underline{0}) < x - \hat{m}; \\ \{0\} & \text{if } x - \hat{m} \leq f(\underline{0}) < x \text{ and } l(f(\underline{0})) < c_1(\hat{m}) + l(x - \hat{m}); \\ \{\hat{m}\} & \text{if } x - \hat{m} \leq f(\underline{0}) < x \text{ and } l(f(\underline{0})) > c_1(\hat{m}) + l(x - \hat{m}); \\ \{\hat{m}\} & \text{if } x \leq f(\underline{0}). \end{cases}$$

$$BR_1(\hat{r}) = \begin{cases} \{0\} & \text{if } f(\hat{r}) < x - \hat{m}; \\ \{0\} & \text{if } x - \hat{m} \leq f(\hat{r}) < x \text{ and } l(f(\hat{r})) < c_1(\hat{m}) + l(x - \hat{m}); \\ \{\hat{m}\} & \text{if } x - \hat{m} \leq f(\hat{r}) < x \text{ and } l(f(\hat{r})) > c_1(\hat{m}) + l(x - \hat{m}); \\ \{\hat{m}\} & \text{if } x \leq f(\hat{r}). \end{cases}$$

From Lemma 2, player 2's best responses are as follows:

$$BR_2(0) = \begin{cases} \{\underline{0}\} & \text{if } x \leq f(\hat{r}); \\ \{\underline{0}\} & \text{if } f(\hat{r}) < x \text{ and } l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r})); \\ \hat{R} & \text{if } f(\hat{r}) < x \text{ and } l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r})). \end{cases}$$

$$BR_2(\hat{m}) = \begin{cases} \{\underline{0}\} & \text{if } x - \hat{m} \leq f(\hat{r}); \\ \{\underline{0}\} & \text{if } f(\hat{r}) < x - \hat{m} \text{ and } l(x - \hat{m}) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r})); \\ \hat{R} & \text{if } f(\hat{r}) < x - \hat{m} \text{ and } l(x - \hat{m}) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r})). \end{cases}$$

Propositions 4 through 9 characterize equilibria in my model.

Proof of Proposition 4. Because $f(\underline{0}) < x - \hat{m}$, $BR_1(\underline{0}) = \{0\}$. Because $f(\hat{r}) < f(\underline{0})$ and $f(\underline{0}) < x - \hat{m}$, we have $f(\hat{r}) < x - \hat{m}$ and $BR_1(\hat{r}) = \{0\}$. Because $f(\hat{r}) < x - \hat{m}$, $f(\hat{r}) < x$. As in (i), if $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, then $BR_2(0) = \{\underline{0}\}$, and $(0, \underline{0})$ is the only equilibrium. As in (ii), if $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, then $BR_2(0) = \hat{R}$, and $(0, \hat{r})$ is the only equilibrium. □

Proof of Proposition 5. (i) Because $x - \hat{m} \leq f(\underline{0}) < x$ and $l(f(\underline{0})) < c_1(\hat{m}) + l(x - \hat{m})$, $BR_1(\underline{0}) = \{0\}$. Because $f(\hat{r}) < x - \hat{m}$, $BR_1(\hat{r}) = \{0\}$. Because $f(\hat{r}) < x - \hat{m}$, $f(\hat{r}) < x$. Because $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, $BR_2(0) = \{\underline{0}\}$. Thus, $(0, \underline{0})$ is the only equilibrium.

(ii) Because $x - \hat{m} \leq f(\underline{0}) < x$ and $l(f(\underline{0})) < c_1(\hat{m}) + l(x - \hat{m})$, $BR_1(\underline{0}) = \{0\}$. Because $f(\hat{r}) < x - \hat{m}$, $BR_1(\hat{r}) = \{0\}$. Because $f(\hat{r}) < x - \hat{m}$, $f(\hat{r}) < x$. Also, because $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, $BR_2(0) = \hat{R}$. Thus, $(0, \hat{r})$ is the only equilibrium.

(iii) Because $f(\hat{r}) < x - \hat{m}$, $f(\hat{r}) < x$. Also, because $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, $BR_2(0) = \{\underline{0}\}$. Because $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, $l(x - \hat{m}) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$. Also, because $f(\hat{r}) < x - \hat{m}$, $BR_2(\hat{m}) = \{\underline{0}\}$. Because $x - \hat{m} \leq f(\underline{0}) < x$ and $l(f(\underline{0})) > c_1(\hat{m}) + l(x - \hat{m})$, $BR_1(\underline{0}) = \{\hat{m}\}$. Thus, $(\hat{m}, \underline{0})$ is the only equilibrium.

(iv) Because $f(\hat{r}) < x - \hat{m}$, $BR_1(\hat{r}) = \{0\}$. Because $f(\hat{r}) < x - \hat{m}$, $f(\hat{r}) < x$. Also, because $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, $BR_2(0) = \hat{R}$. Because $x - \hat{m} \leq f(\underline{0}) < x$ and $l(f(\underline{0})) > c_1(\hat{m}) + l(x - \hat{m})$, $BR_1(\underline{0}) = \{\hat{m}\}$. Because $f(\hat{r}) < x - \hat{m}$ and $l(x - \hat{m}) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, $BR_2(\hat{m}) = \{\underline{0}\}$. Thus, $(0, \hat{r})$ and $(\hat{m}, \underline{0})$ are the only pure strategy equilibria.

(v) Because $f(\hat{r}) < x - \hat{m}$, $f(\hat{r}) < x$. Because $l(x - \hat{m}) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$ and $BR_2(0) = \hat{R}$. Because $f(\hat{r}) < x - \hat{m}$ and $l(x - \hat{m}) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, $BR_2(\hat{m}) = \hat{R}$. Because $f(\hat{r}) < x - \hat{m}$, $BR_1(\hat{r}) = \{0\}$. The only equilibrium is $(0, \hat{r})$. $\square$

Proof of Proposition 6. Because $x - \hat{m} \leq f(\hat{r})$ and $f(\underline{0}) < x$, we have $x - \hat{m} \leq f(\underline{0}) < x$ and $x - \hat{m} \leq f(\hat{r}) < x$.

(i) Because $x - \hat{m} \leq f(\underline{0}) < x$ and $l(f(\underline{0})) < c_1(\hat{m}) + l(x - \hat{m})$, $BR_1(\underline{0}) = \{0\}$. Because $l(f(\underline{0})) < c_1(\hat{m}) + l(x - \hat{m})$, $l(f(\hat{r})) < c_1(\hat{m}) + l(x - \hat{m})$. Also, because $x - \hat{m} \leq f(\hat{r}) < x$, $BR_1(\hat{r}) = \{0\}$. Because $f(\hat{r}) < x$ and $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, $BR_2(0) = \{\underline{0}\}$. Thus, $(0, \underline{0})$ is the only equilibrium.

(ii) Because $f(\hat{r}) < x$ and $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, $BR_2(0) = \{\underline{0}\}$. Because $x - \hat{m} \leq f(\hat{r})$, $BR_2(\hat{m}) = \{\underline{0}\}$. Because $x - \hat{m} \leq f(\underline{0}) < x$ and $l(f(\underline{0})) > c_1(\hat{m}) + l(x - \hat{m})$, $BR_1(\underline{0}) = \{\hat{m}\}$. Thus, $(\hat{m}, \underline{0})$ is the only equilibrium.

(iii) Because $x - \hat{m} \leq f(\underline{0}) < x$ and $l(f(\underline{0})) < c_1(\hat{m}) + l(x - \hat{m})$, $BR_1(\underline{0}) = \{0\}$. Because $l(f(\underline{0})) < c_1(\hat{m}) + l(x - \hat{m})$, $l(f(\hat{r})) < c_1(\hat{m}) + l(x - \hat{m})$. Also, because $x - \hat{m} \leq f(\hat{r}) < x$, $BR_1(\hat{r}) = \{0\}$. Because $f(\hat{r}) < x$ and $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, $BR_2(0) = \hat{R}$. Thus, $(0, \hat{r})$ is the only equilibrium.

(iv) Because $x - \hat{m} \leq f(\hat{r}) < x$ and $l(f(\hat{r})) < c_1(\hat{m}) + l(x - \hat{m})$, $BR_1(\hat{r}) = \{0\}$. Because $f(\hat{r}) < x$ and $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, $BR_2(0) = \hat{R}$. Because $x - \hat{m} \leq f(\underline{0}) < x$ and $l(f(\underline{0})) > c_1(\hat{m}) + l(x - \hat{m})$, $BR_1(\underline{0}) = \{\hat{m}\}$. Because $x - \hat{m} \leq f(\hat{r})$, $BR_2(\hat{m}) = \{\underline{0}\}$. Thus, $(0, \hat{r})$ and $(\hat{m}, \underline{0})$ are the only pure strategy equilibria.

(v) Because $l(f(\hat{r})) > c_1(\hat{m}) + l(x - \hat{m})$, $l(f(\underline{0})) > c_1(\hat{m}) + l(x - \hat{m})$. Also, because $x - \hat{m} \leq f(\underline{0}) < x$, $BR_1(\underline{0}) = \{\hat{m}\}$. Because $x - \hat{m} \leq f(\hat{r}) < x$ and $l(f(\hat{r})) > c_1(\hat{m}) + l(x - \hat{m})$, $BR_1(\hat{r}) = \{\hat{m}\}$. Because $x - \hat{m} \leq f(\hat{r})$, $BR_2(\hat{m}) = \{\underline{0}\}$. The only equilibrium is $(\hat{m}, \underline{0})$. $\square$

Proof of Proposition 7. (i) Because $f(\hat{r}) < x - \hat{m}$, $f(\hat{r}) < x$. Also, because $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, $BR_2(0) = \{\underline{0}\}$. Because $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, $l(x - \hat{m}) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$. Also, because $f(\hat{r}) < x - \hat{m}$, $BR_2(\hat{m}) = \{\underline{0}\}$. Because $x \leq f(\underline{0})$, $BR_1(\underline{0}) = \{\hat{m}\}$. Thus, $(\hat{m}, \underline{0})$ is the only equilibrium.

(ii) Because $f(\hat{r}) < x - \hat{m}$, $BR_1(\hat{r}) = \{0\}$. Because $f(\hat{r}) < x - \hat{m}$, $f(\hat{r}) < x$. Also, because $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, $BR_2(0) = \hat{R}$. Because $x \leq f(\underline{0})$, $BR_1(\underline{0}) = \{\hat{m}\}$. Because $f(\hat{r}) < x - \hat{m}$ and $l(x - \hat{m}) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, $BR_2(\hat{m}) = \{\underline{0}\}$. Thus, $(0, \hat{r})$ and $(\hat{m}, \underline{0})$ are the only pure strategy equilibria.

(iii) Because $l(x - \hat{m}) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$. Also, because $f(\hat{r}) < x$, $BR_2(0) = \hat{R}$. Because $f(\hat{r}) < x - \hat{m}$ and $l(x - \hat{m}) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, $BR_2(\hat{m}) = \hat{R}$. Because $f(\hat{r}) < x - \hat{m}$, $BR_1(\hat{r}) = \{0\}$. Thus, $(0, \hat{r})$ is the only equilibrium. $\square$

Proof of Proposition 8. (i) Because $f(\hat{r}) < x$ and $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, $BR_2(0) = \{\underline{0}\}$. Because $x - \hat{m} \leq f(\hat{r})$, $BR_2(\hat{m}) = \{\underline{0}\}$. Because $x \leq f(\underline{0})$, $BR_1(\underline{0}) = \{\hat{m}\}$. Thus, $(\hat{m}, \underline{0})$ is the only equilibrium.

(ii) Because $x \leq f(\underline{0})$, $BR_1(\underline{0}) = \{\hat{m}\}$. Because $x - \hat{m} \leq f(\hat{r}) < x$ and $l(f(\hat{r})) > c_1(\hat{m}) + l(x - \hat{m})$, $BR_1(\hat{r}) = \{\hat{m}\}$. Because $x - \hat{m} \leq f(\hat{r})$, $BR_2(\hat{m}) = \{\underline{0}\}$. Thus, $(\hat{m}, \underline{0})$ is the only equilibrium.

(iii) Because $x - \hat{m} \leq f(\hat{r}) < x$ and $l(f(\hat{r})) < c_1(\hat{m}) + l(x - \hat{m})$, $BR_1(\hat{r}) = \{0\}$. Because $f(\hat{r}) < x$ and $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, $BR_2(0) = \hat{R}$. Because $x \leq f(\underline{0})$, $BR_1(\underline{0}) = \{\hat{m}\}$. Because $x - \hat{m} \leq f(\hat{r})$, $BR_2(\hat{m}) = \{\underline{0}\}$. The only pure strategy equilibria are $(0, \hat{r})$ and $(\hat{m}, \underline{0})$. □

Proof of Proposition 9. Because $x \leq f(\hat{r})$, $x < f(\underline{0})$ and $BR_1(\underline{0}) = \{\hat{m}\}$. Because $x \leq f(\hat{r})$, $BR_1(\hat{r}) = \{\hat{m}\}$. Because $x \leq f(\hat{r})$, $x - \hat{m} < f(\hat{r})$ and $BR_2(\hat{m}) = \{\underline{0}\}$. Thus, $(\hat{m}, \underline{0})$ is the only equilibrium. □

Finally I present the proof of Proposition 10.

Proof of Proposition 10. Show that $v(\{1\}) + v(\{2\}) \leq v(\{1, 2\})$. First, suppose that $x - \hat{m} \leq f(\underline{0}) < x$.

From Lemma 1, player 1's best response is 0 or $\hat{m}$. If 0 is a best response to $\underline{0}$, then $v(\{1\}) = u_1(0, \underline{0})$. Because $f(\underline{0}) < x$, we have $u_1(0, \underline{0}) = w - l(f(\underline{0}))$. If $\hat{m}$ is a best response to $\underline{0}$, then $v(\{1\}) = u_1(\hat{m}, \underline{0})$. Because $x - \hat{m} \leq f(\underline{0})$, we have $u_1(\hat{m}, \underline{0}) = w - c_1(\hat{m}) - l(x - \hat{m})$. Thus, $v(\{1\}) = w - l(f(\underline{0}))$ or $v(\{1\}) = w - c_1(\hat{m}) - l(x - \hat{m})$.

From Lemma 2, player 2's best response is $\underline{0}$ or $\hat{r}$. If $\underline{0}$ is a best response to 0, then $v(\{2\}) = u_2(0, \underline{0})$. Because $f(\underline{0}) < x$, we have $u_2(0, \underline{0}) = w - l(f(\underline{0}))$. If $\hat{r}$ is a best response to 0, then $v(\{2\}) = u_2(0, \hat{r})$. Because $f(\hat{r}) < f(\underline{0})$ and $f(\underline{0}) < x$, we have $u_2(0, \hat{r}) = w - c_2(\sum \hat{r}_{ij}) - l(f(\hat{r}))$. Thus, $v(\{2\}) = w - l(f(\underline{0}))$ or $v(\{2\}) = w - c_2(\sum \hat{r}_{ij}) - l(f(\hat{r}))$. Divide into four cases.

Case 1. If $v(\{1\}) = w - l(f(\underline{0}))$ and $v(\{2\}) = w - l(f(\underline{0}))$,

$$v(\{1\}) + v(\{2\}) = 2w - 2l(f(\underline{0})) = U(0, \underline{0}) \leq v(\{1, 2\}).$$

Case 2. If $v(\{1\}) = w - l(f(\underline{0}))$ and $v(\{2\}) = w - c_2(\sum \hat{r}_{ij}) - l(f(\hat{r}))$, because $f(\hat{r}) < f(\underline{0})$ implies $v(\{1\}) < w - l(f(\hat{r}))$,

$$v(\{1\}) + v(\{2\}) < 2w - c_2(\sum \hat{r}_{ij}) - 2l(f(\hat{r})) = U(0, \hat{r}) \leq v(\{1, 2\}).$$

Case 3. If $v(\{1\}) = w - c_1(\hat{m}) - l(x - \hat{m})$ and $v(\{2\}) = w - l(f(\underline{0}))$, because $x - \hat{m} \leq f(\underline{0})$ implies $v(\{2\}) \leq w - l(x - \hat{m})$,

$$v(\{1\}) + v(\{2\}) \leq 2w - c_1(\hat{m}) - 2l(x - \hat{m}) = U(\hat{m}, \underline{0}) \leq v(\{1, 2\}).$$

Case 4. If $v(\{1\}) = w - c_1(\hat{m}) - l(x - \hat{m})$ and $v(\{2\}) = w - c_2(\sum \hat{r}_{ij}) - l(f(\hat{r}))$, consider two subcases. If $x - \hat{m} \leq f(\hat{r})$, then $v(\{2\}) \leq w - l(x - \hat{m})$. Thus,

$$v(\{1\}) + v(\{2\}) \leq 2w - c_1(\hat{m}) - 2l(x - \hat{m}) = U(\hat{m}, \underline{0}) \leq v(\{1, 2\}).$$

However, if $f(\hat{r}) \leq x - \hat{m}$, then $v(\{1\}) \leq w - l(f(\hat{r}))$. Thus,

$$v(\{1\}) + v(\{2\}) \leq 2w - c_2(\sum \hat{r}_{ij}) - 2l(f(\hat{r})) = U(0, \hat{r}) \leq v(\{1,2\}).$$

Second, suppose that $f(\underline{0}) < x - \hat{m}$. From Lemma 1, 0 is the only best response to $\underline{0}$, and $v(\{1\}) = u_1(0, \underline{0})$. Because $f(\underline{0}) < x - \hat{m}$ and $x - \hat{m} < x$, $u_1(0, \underline{0}) = w - l(f(\underline{0}))$ and $v(\{1\}) = w - l(f(\underline{0}))$.

From Lemma 2, player 2's best response is $\underline{0}$ or $\hat{r}$. If $\underline{0}$ is a best response to 0, then $v(\{2\}) = u_2(0, \underline{0})$. Because $f(\underline{0}) < x$, we have $u_2(0, \underline{0}) = w - l(f(\underline{0}))$. Thus, $v(\{2\}) = w - l(f(\underline{0}))$. As in Case 1, we have $v(\{1\}) + v(\{2\}) \leq v(\{1,2\})$.

However, if $\hat{r}$ is a best response to 0, then $v(\{2\}) = u_2(0, \hat{r})$. Also, if $\hat{r}$ is a best response to 0, then $f(\hat{r}) < x$ and $u_2(0, \hat{r}) = w - c_2(\sum \hat{r}_{ij}) - l(f(\hat{r}))$. Thus, $v(\{2\}) = w - c_2(\sum \hat{r}_{ij}) - l(f(\hat{r}))$. As in Case 2, we have $v(\{1\}) + v(\{2\}) \leq v(\{1,2\})$.

Third, suppose that $x \leq f(\underline{0})$. From Lemma 1, $\hat{m}$ is the only best response to $\underline{0}$, and $v(\{1\}) = u_1(\hat{m}, \underline{0})$. Because $x - \hat{m} < x$ and $x \leq f(\underline{0})$, $u_1(\hat{m}, \underline{0}) = w - c_1(\hat{m}) - l(x - \hat{m})$ and $v(\{1\}) = w - c_1(\hat{m}) - l(x - \hat{m})$.

From Lemma 2, player 2's best response is $\underline{0}$ or $\hat{r}$. If $\underline{0}$ is a best response to 0, then $v(\{2\}) = u_2(0, \underline{0})$. Because $x \leq f(\underline{0})$, we have $u_2(0, \underline{0}) = w - l(x)$. Thus, $v(\{2\}) = w - l(x)$. Because $x - \hat{m} < x$, we have $v(\{2\}) < w - l(x - \hat{m})$. Thus, as in Case 3, we have $v(\{1\}) + v(\{2\}) < 2w - c_1(\hat{m}) - 2l(x - \hat{m}) \leq v(\{1,2\})$.

However, if $\hat{r}$ is a best response to 0, then $v(\{2\}) = u_2(0, \hat{r})$. Also, if $\hat{r}$ is a best response to 0, then $f(\hat{r}) < x$ and $u_2(0, \hat{r}) = w - c_2(\sum \hat{r}_{ij}) - l(f(\hat{r}))$. Thus, $v(\{2\}) = w - c_2(\sum \hat{r}_{ij}) - l(f(\hat{r}))$. As in Case 4, we have $v(\{1\}) + v(\{2\}) \leq v(\{1,2\})$.

Therefore, v is superadditive. □

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